q-Integration on quantum spaces

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Abstract. In this article we present explicit formulae for q-integration on quantum spaces which could be of particular importance in physics, i.e., q-deformed Minkowski space and q-deformed Euclidean space in three or four dimensions. Furthermore, our formulae can be regarded as a generalization of Jackson's q-integral to three or four dimensions and provide a new possibility for an integration over the whole space being invariant under translations and rotations.

1 Introduction

The history of the natural sciences shows us that revolutionary changes in our understanding of physical phenomena have often been accompanied by new developments in mathematics. Newtonian mechanics is one famous example for such a situation in physics, as its general formulation would not have been possible without the ideas of differential calculus. Due to this fact one may believe that a new theory giving a more detailed description of nature has to be based on a modified version of traditional mathematics.

Quantum spaces, which are defined as co-module algebras of quantum groups and which can be interpreted as deformations of ordinary co-ordinate algebras [1], could provide a proper framework for developing such a modification [2,3]. For our purposes it is sufficient to consider a quantum space as an algebra $\mathcal{A}_q$ of a formal power series in the non-commuting co-ordinates $X_1, X_2, \ldots, X_n$:

$$\mathcal{A}_q = \mathbb{C}\left[\left[X_1, \ldots X_n\right]\right] / \mathcal{I}, \tag{1}$$

where $\mathcal{I}$ denotes the ideal generated by the relations of the non-commuting co-ordinates.

The algebra $\mathcal{A}_q$ satisfies the Poincaré–Birkhoff–Witt property, i.e. the dimension of the subspace of homogeneous polynomials should be the same as for the commuting co-ordinates. This property is the deeper reason why the monomials of normal ordering $X_1 X_2 \ldots X_n$ constitute a basis of $\mathcal{A}_q$. In particular, we can establish a vector space isomorphism between $\mathcal{A}_q$ and the commutative algebra $\mathcal{A}$ generated by ordinary co-ordinates $x_1, x_2, \ldots, x_n$:

$$\mathcal{W} \, : \, \mathcal{A} \longrightarrow \mathcal{A}_q, \tag{2}$$

$$\mathcal{W}(x_1^{i_1} \ldots x_n^{i_n}) = X_1^{i_1} \ldots X_n^{i_n}.$$

This vector space isomorphism can be extended to an algebra isomorphism introducing a non-commutative product

in $\mathcal{A}$, the so-called $\star$-product [4,5]. This product is defined by the relation

$$\mathcal{W}(f \star g) = \mathcal{W}(f) \cdot \mathcal{W}(g), \tag{3}$$

where f and g are formal power series in $\mathcal{A}$. In [6] we have calculated the $\star$-product for quantum spaces which could be of particular importance in physics, i.e. q-deformed Minkowski space and q-deformed Euclidean space in three or four dimensions.

Additionally, for each of these quantum spaces there exists a symmetry algebra [7,8] and a covariant differential calculus [9] which can provide an action upon the quantum spaces under consideration. By means of the relation

$$\mathcal{W}(h \triangleright f) := h \triangleright \mathcal{W}(f), \quad h \in \mathcal{H}, \quad f \in \mathcal{A}, \tag{4}$$

we are also able to introduce an action upon the corresponding commutative algebra. In our previous work [10] we have presented explicit formulae for such representations.

In the following it is our aim to derive some sort of q-integration on commutative algebras which are, via the algebra isomorphism $\mathcal{W}$, related to q-deformed Minkowski space or q-deformed Euclidean space in three and four dimensions. In some sense our considerations can be supposed to be a generalization of the celebrated Jackson integral [11] to higher dimensions. To this end we will start with introducing new elements which are formally inverse to the partial derivatives of the given covariant differential calculi. Such an extension of the algebra of partial derivatives will lead to additional commutation relations. Finally, the representations of the partial derivatives given in [10] will aid us in identifying these new elements with particular solutions to some q-difference equations. In this way, we can interpret our results as a method to discretize classical integrals of more than one dimension.

Furthermore, we will see that we have to distinguish between left and right integrals. For both types formulae

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for integration by parts can be derived. It is also possible to define volume integrals which are invariant under translations or the action of symmetry generators if surface terms are neglected. Surprisingly, these integrals obey a rather simple structure, as the single integrations for the different directions become independent from each other if we integrate over the entire space.

2 q-Deformed Euclidean space in three dimensions

From [12] we know that the partial derivatives ∂^+, ∂^3, ∂^- satisfy the same relations as the quantum space coordinates X^+, X^3, X^-. Thus we have

$$\partial^3 \partial^+ = q^2 \partial^+ \partial^3, \qquad \partial^- \partial^3 = q^2 \partial^3 \partial^-,$$
$$\partial^- \partial^+ = \partial^+ \partial^- + \lambda (\partial^3)^2, \tag{5}$$

where $\lambda = q - q^{-1}$ and $q > 1$. Now, we would like to extend the algebra of partial derivatives by inverse elements, such that

$$\partial^+ (\partial^+)^{-1} = (\partial^+)^{-1} \partial^+ = 1,$$
$$\partial^3 (\partial^3)^{-1} = (\partial^3)^{-1} \partial^3 = 1, \tag{6}$$
$$\partial^- (\partial^-)^{-1} = (\partial^-)^{-1} \partial^- = 1.$$

It can easily be shown that these relations imply the formulae

$$(\partial^3)^{-1} \partial^+ = q^{-2} \partial^+ (\partial^3)^{-1}, \quad \partial^3 (\partial^+)^{-1} = q^{-2} (\partial^+)^{-1} \partial^3,$$
$$(\partial^-)^{-1} \partial^3 = q^{-2} \partial^3 (\partial^-)^{-1}, \quad \partial^- (\partial^3)^{-1} = q^{-2} (\partial^3)^{-1} \partial^-,$$
$$(\partial^-)^{-1} \partial^+ = \partial^+ (\partial^-)^{-1} - q^{-4} \lambda (\partial^3)^2 (\partial^-)^{-2}, \tag{7}$$
$$\partial^- (\partial^+)^{-1} = (\partial^+)^{-1} \partial^- - q^{-4} \lambda (\partial^+)^{-2} (\partial^3)^2.$$

In addition, we can also find the following identities:

$$(\partial^3)^{-1} (\partial^+)^{-1} = q^2 (\partial^+)^{-1} (\partial^3)^{-1}, \tag{8}$$
$$(\partial^-)^{-1} (\partial^3)^{-1} = q^2 (\partial^3)^{-1} (\partial^-)^{-1},$$
$$(\partial^-)^{-1} (\partial^+)^{-1} = \sum_{i=0}^{\infty} \lambda^i \, [[i]]_{q^4}! \left(\begin{bmatrix} -1 \\ i \end{bmatrix}_{q^4} \right)^2$$
$$\cdot (\partial^+)^{-(i+1)} (\partial^3)^{2i} (\partial^-)^{-(i+1)}.$$

From the commutation relations between symmetry generators and partial derivatives [12] we find the expressions

$$L^+ (\partial^+)^{-1} = (\partial^+)^{-1} L^+, \tag{9}$$
$$L^+ (\partial^3)^{-1} = (\partial^3)^{-1} L^+ + q^{-1} \partial^+ (\partial^3)^{-2} \tau^{-\frac{1}{2}},$$
$$L^+ (\partial^-)^{-1} = (\partial^-)^{-1} L^+ + q^{-1} \partial^3 (\partial^-)^{-2} \tau^{-\frac{1}{2}},$$
$$L^- (\partial^-)^{-1} = (\partial^-)^{-1} L^-, \tag{10}$$

$$L^- (\partial^3)^{-1} = (\partial^3)^{-1} L^- - q^{-3} (\partial^3)^{-2} \partial^- \tau^{-\frac{1}{2}},$$
$$L^- (\partial^+)^{-1} = (\partial^+)^{-1} L^- - q^{-4} (\partial^+)^{-2} \partial^3 \tau^{-\frac{1}{2}},$$

$$\tau^{-\frac{1}{2}} (\partial^+)^{-1} = q^{-2} (\partial^+)^{-1} \tau^{-\frac{1}{2}}, \tag{11}$$
$$\tau^{-\frac{1}{2}} (\partial^-)^{-1} = q^2 (\partial^-)^{-1} \tau^{-\frac{1}{2}},$$
$$\tau^{-\frac{1}{2}} (\partial^3)^{-1} = (\partial^3)^{-1} \tau^{-\frac{1}{2}},$$

$$\Lambda^{\frac{1}{2}} (\partial^A)^{-1} = q^2 (\partial^A)^{-1} \Lambda^{\frac{1}{2}}, \quad A = \pm, 3. \tag{12}$$

Applying the substitutions

$$\partial^A \to \hat{\partial}^A, \quad (\partial^A)^{-1} \to (\hat{\partial}^A)^{-1}, \quad A = \pm, 3 \tag{13}$$

to all expressions presented so far we get the corresponding relations of the second differential calculus (generated by the conjugated partial derivatives $\hat{\partial}^A$ [12]).

In [10] it was shown that according to

$$\partial^A \triangleright F = \left((\partial^A_{(i=0)}) + (\partial^A_{(i>0)}) \right) F \tag{14}$$

the representations of our partial derivatives can be divided into a classical part and corrections vanishing in the undeformed limit $q \to 1$. Thus, seeking a solution to the equation

$$\partial^A \triangleright F = f \tag{15}$$

for given f it is reasonable to consider the following expression:

$$F = (\partial^A)^{-1} \triangleright f = \frac{1}{(\partial^A_{(i=0)}) + (\partial^A_{(i>0)})} f \tag{16}$$

$$= \frac{1}{(\partial^A_{(i=0)}) \left(1 + (\partial^A_{(i=0)})^{-1} (\partial^A_{(i>0)}) \right)} f$$

$$= \frac{1}{1 + (\partial^A_{(i=0)})^{-1} (\partial^A_{(i>0)})} \cdot \frac{1}{(\partial^A_{(i=0)})} f$$

$$= \sum_{k=0}^{\infty} (-1)^k \left[(\partial^A_{(i=0)})^{-1} (\partial^A_{(i>0)}) \right]^k (\partial^A_{(i=0)})^{-1} f.$$

To apply this formula, we need to identify the contributions $\partial^A_{(i=0)}$ and $\partial^A_{(i>0)}$ the representations of our partial derivatives consist of. However, their explicit form can be read off quite easily from the results in [10]. Hence we have

$$(\partial^+_{(i=0)}) f = -q D^-_{q^4} f(q^2 x^3), \tag{17}$$
$$(\partial^3_{(i=0)}) f = D^3_{q^2} f(q^2 x^+),$$
$$(\partial^-_{(i=0)}) f = -q^{-1} D^+_{q^4} f$$

and

$$(\partial^+_{(i>0)}) f = -q \lambda x^+ (D^3_{q^2})^2 f, \tag{18}$$
$$(\partial^3_{(i>0)}) f = 0,$$

$$(\partial^-_{(i>0)})f = 0.$$

Furthermore, it is easily seen that the inverse operators $(\partial^A_{(i=0)})^{-1}$ are given by

$$(\partial^+_{(i=0)})^{-1}f = -q^{-1}(D_{q^4}^-)^{-1}f(q^{-2}x^3), \qquad (19)$$

$$(\partial^3_{(i=0)})^{-1}f = (D_{q^2}^3)^{-1}f(q^{-2}x^+),$$

$$(\partial^-_{(i=0)})^{-1}f = -q(D_{q^4}^+)^{-1}f,$$

where $D_{q^a}^A$ and $(D_{q^a}^A)^{-1}$ denote Jackson derivatives and Jackson integrals, respectively. Inserting these expressions into formula (16) we can represent the inverse counterparts of the partial derivatives in the form

$$(\partial^-)^{-1} \triangleright f = -q(D_{q^4}^+)^{-1}f, \qquad (20)$$

$$(\partial^3)^{-1} \triangleright f = (D_{q^2}^3)^{-1}f(q^{-2}x^+),$$

$$(\partial^+)^{-1} \triangleright f = -q^{-1}\sum_{k=0}^{\infty}(-\lambda)^k q^{2k(k+1)}\left[x^+(D_{q^4}^-)^{-1}(D_{q^2}^3)^2\right]^k$$

$$\cdot (D_{q^4}^-)^{-1}f(q^{-2(k+1)}x^3).$$

Repeating the identical steps as before, we can also find solutions to the equations

$$\hat{\partial}^A \tilde{\triangleright} F = f, \qquad (21)$$

$$F \tilde{\triangleleft} \partial^A = f,$$

$$F \triangleleft \hat{\partial}^A = f,$$

where the explicit form of the action of our partial derivatives is again taken from [10]. However, in [10] it was also shown that the different representations of our partial derivatives can be transformed into each other by a straightforward application of so-called crossing symmetries. Due to this fact the solutions to (21) are easily obtained from (20), if we apply the transformations

$$(\partial^\pm)^{-1} \triangleright f \overset{\pm \to \mp}{\underset{q \to 1/q}{\longleftrightarrow}} (\hat{\partial}^\mp)^{-1}\tilde{\triangleright}f, \qquad (22)$$

$$(\partial^3)^{-1} \triangleright f \overset{\pm \to \mp}{\underset{q \to 1/q}{\longleftrightarrow}} (\hat{\partial}^3)^{-1}\tilde{\triangleright}f,$$

which concretely mean that the expressions on the right and left hand side are related to each other by the substitutions[1]

$$x^\pm \to x^\mp, \quad q^{\pm 1} \to q^{\mp 1}, \quad \hat{n}^\pm \to -\hat{n}^\mp \qquad (23)$$

$$D_{q^a}^\pm \to D_{q^{-a}}^\mp, \quad (D_{q^a}^\pm)^{-1} \to (D_{q^{-a}}^\mp)^{-1}.$$

In the same way we have

$$f \tilde{\triangleleft} (\partial^\pm)^{-1} \overset{+\leftrightarrow -}{\longleftrightarrow} -q^6(\hat{\partial}^\mp)^{-1}\tilde{\triangleright}f, \qquad (24)$$

$$f \tilde{\triangleleft} (\partial^3)^{-1} \overset{+\leftrightarrow -}{\longleftrightarrow} -q^6(\hat{\partial}^3)^{-1}\tilde{\triangleright}f,$$

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[1] For the notation see Appendix A, please

$$f \triangleleft (\hat{\partial}^\pm)^{-1} \overset{+\leftrightarrow -}{\longleftrightarrow} -q^{-6}(\partial^\mp)^{-1} \triangleright f, \qquad (25)$$

$$f \triangleleft (\hat{\partial}^3)^{-1} \overset{+\leftrightarrow -}{\longleftrightarrow} -q^{-6}(\partial^3)^{-1} \triangleright f,$$

which symbolizes a transition described by the substitutions

$$x^\pm \to x^\mp, \; D_{q^a}^\pm \to D_{q^a}^\mp, \; (D_{q^a}^\pm)^{-1} \to (D_{q^a}^\mp)^{-1}, \; \hat{n}^\pm \to \hat{n}^\mp. \qquad (26)$$

Next, we would like to have a closer look at the question in which sense the operator $(\partial^A)^{-1}$ inverse to the operator ∂^A is. First of all, we require that all lower limits of Jackson integrals appearing in the representations of $(\partial^A)^{-1}$ are set equal to zero. In this case one can easily verify the identities

$$D_{q^a}^A\left[(D_{q^a}^A)^{-1} f|_0^{x^A}\right] = f, \qquad (27)$$

$$(D_{q^a}^A)^{-1}D_{q^a}^A f|_0^{x^A} = f|_0^{x^A},$$

leading to

$$\partial^A_{(i=0)}\left[(\partial^A_{(i=0)})^{-1} f|_0^{x^{\bar{A}}}\right] = f, \qquad (28)$$

$$(\partial^A_{(i=0)})^{-1}\partial^A_{(i=0)} f|_0^{x^{\bar{A}}} = f|_0^{x^{\bar{A}}}.$$

It is important to realize that in the above formulae the coordinates giving the upper limits of integration have to be labeled by different indices than the corresponding operators $(\partial^A_{(i=0)})^{-1}$. Let us now introduce the operator

$$\hat{C}^A f = -(\partial^A_{(i=0)})^{-1}(\partial^A_{(i>0)}) f|_0^{x^{\bar{A}}}. \qquad (29)$$

With the identities (14), (16) and (28) one can then compute that

$$(\partial^A)^{-1}\partial^A f\big|_0^{x^{\bar{A}}} \qquad (30)$$

$$= \sum_{k=0}^{\infty}(\hat{C}^A)^k \left(f - f(x^{\bar{A}} = 0)\right)\Big|_0^{x^{\bar{A}}} - \sum_{k=1}^{\infty}(\hat{C}^A)^k f\Big|_0^{x^{\bar{A}}}$$

$$= f - f(x^{\bar{A}} = 0) - \sum_{k=1}^{\infty}(\hat{C}^A)^k f(x^{\bar{A}} = 0)\Big|_0^{x^{\bar{A}}},$$

where the limits of the integration intervals *always refer to the Jackson integrals* in the expressions of $(\partial^A)^{-1}$ or $\hat{C}^A$. In addition to this one readily checks that

$$\partial^A\left[(\partial^A)^{-1} f|_0^{x^{\bar{A}}}\right] = f, \qquad (31)$$

as the representations of $(\partial^A)^{-1}$ have been determined in such a way that they give solutions to (15).

As a next step we wish to present formulae for integration by parts. Before doing this let us collect some notation that will be used in the following. In the case of right and left derivatives we use the abbreviations

$$\partial^A_{\mathrm{L}}f \equiv \partial^A \triangleright f, \qquad (32)$$

$$\partial^A_{\mathrm{R}}f \equiv f \tilde{\triangleleft} \partial^A,$$

$$\hat{\partial}_{\mathrm{L}}^A f \equiv \hat{\partial}^A \tilde{\triangleright} f, \tag{33}$$

$$\hat{\partial}_{\mathrm{R}}^A f \equiv f \triangleleft \hat{\partial}^A.$$

Likewise for right and left integrals we abbreviate

$$(\partial^A)_{\mathrm{L}}^{-1} f \equiv (\partial^A)^{-1} \triangleright f, \tag{34}$$

$$(\partial^A)_{\mathrm{R}}^{-1} f \equiv f \tilde{\triangleleft} (\partial^A)^{-1},$$

$$(\hat{\partial}^A)_{\mathrm{L}}^{-1} f \equiv (\hat{\partial}^A)^{-1} \tilde{\triangleright} f, \tag{35}$$

$$(\hat{\partial}^A)_{\mathrm{R}}^{-1} f \equiv f \triangleleft (\hat{\partial}^A)^{-1}.$$

Furthermore, we set

$$f\|_0^{x^{\bar{A}}} \equiv f - f(x^{\bar{A}} = 0) - \sum_{k=1}^{\infty} (\hat{C}^A)^k \, f(x^{\bar{A}} = 0)\Big|_0^{x^{\bar{A}}}. \tag{36}$$

Now, we are in a position to write down rules for integration by parts. In complete analogy to the classical case these rules can be derived from the Leibniz rules of the corresponding partial derivatives [10]. In this way we get

$$(\partial^-)_{\mathrm{L}}^{-1} (\partial_{\mathrm{L}}^- f) \star g\big|_{x^+=0}^a \tag{37}$$

$$= f \star g \,\|_{x^+=0}^a - (\partial^-)_{\mathrm{L}}^{-1} (\Lambda^{1/2}\tau^{-1/2}f) \star \partial_{\mathrm{L}}^- g\Big|_{x^+=0}^a,$$

$$(\partial^3)_{\mathrm{L}}^{-1} (\partial_{\mathrm{L}}^3 f) \star g\big|_{x^3=0}^a \tag{38}$$

$$= f \star g \,\|_{x^3=0}^a - (\partial^3)_{\mathrm{L}}^{-1} (\Lambda^{1/2}f) \star \partial_{\mathrm{L}}^3 g\Big|_{x^3=0}^a$$

$$- \lambda\lambda_+(\partial^3)_{\mathrm{L}}^{-1} (\Lambda^{1/2}L^+ f) \star \partial_{\mathrm{L}}^- g\Big|_{x^3=0}^a,$$

$$(\partial^+)_{\mathrm{L}}^{-1} (\partial_{\mathrm{L}}^+ f) \star g\big|_{x^-=0}^a \tag{39}$$

$$= f \star g \,\|_{x^-=0}^a - (\partial^+)_{\mathrm{L}}^{-1} (\Lambda^{1/2}\tau^{1/2}f) \star \partial_{\mathrm{L}}^+ g\Big|_{x^-=0}^a$$

$$- q\lambda\lambda_+(\partial^+)_{\mathrm{L}}^{-1} (\Lambda^{1/2}\tau^{1/2}L^+ f) \star \partial_{\mathrm{L}}^3 g\Big|_{x^-=0}^a$$

$$- q^2\lambda^2\lambda_+(\partial^+)_{\mathrm{L}}^{-1} (\Lambda^{1/2}\tau^{1/2}(L^+)^2 f) \star \partial_{\mathrm{L}}^- g\Big|_{x^-=0}^a$$

and

$$(\hat{\partial}^+)_{\mathrm{L}}^{-1} (\hat{\partial}_{\mathrm{L}}^+ f) \star g\Big|_{x^-=0}^a \tag{40}$$

$$= f \star g \,\|_{x^-=0}^a - (\hat{\partial}^+)_{\mathrm{L}}^{-1} (\Lambda^{-1/2}\tau^{-1/2}f) \star \hat{\partial}_{\mathrm{L}}^+ g\Big|_{x^-=0}^a,$$

$$(\hat{\partial}^3)_{\mathrm{L}}^{-1} (\hat{\partial}_{\mathrm{L}}^3 f) \star g\Big|_{x^3=0}^a \tag{41}$$

$$= f \star g \,\|_{x^3=0}^a - (\hat{\partial}^3)_{\mathrm{L}}^{-1} (\Lambda^{-1/2}f) \star \hat{\partial}_{\mathrm{L}}^3 g\Big|_{x^3=0}^a$$

$$- \lambda\lambda_+(\hat{\partial}^3)_{\mathrm{L}}^{-1} (\Lambda^{-1/2}L^- f) \star \hat{\partial}_{\mathrm{L}}^+ g\Big|_{x^3=0}^a,$$

$$(\hat{\partial}^-)_{\mathrm{L}}^{-1} (\hat{\partial}_{\mathrm{L}}^- f) \star g\Big|_{x^+=0}^a \tag{42}$$

$$= f \star g \,\|_{x^+=0}^a - (\hat{\partial}^-)_{\mathrm{L}}^{-1} (\Lambda^{-1/2}\tau^{1/2}f) \star \hat{\partial}_{\mathrm{L}}^- g\Big|_{x^+=0}^a.$$

$$- q^{-1}\lambda\lambda_+ (\hat{\partial}^-)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}\tau^{1/2}L^- f) \star \hat{\partial}_{\mathrm{L}}^3 g\Big|_{x^+=0}^a$$

$$- q^{-2}\lambda^2\lambda_+(\hat{\partial}^-)_{\mathrm{L}}^{-1} (\Lambda^{-1/2}\tau^{1/2}(L^-)^2 f) \star \hat{\partial}_{\mathrm{L}}^+ g\Big|_{x^+=0}^a.$$

In the case of right integrals, however, we have to consider Leibniz rules for right derivatives [10]. Hence, we have

$$(\partial^-)_{\mathrm{R}}^{-1} f \star \partial_{\mathrm{R}}^- g\big|_{x^+=0}^a \tag{43}$$

$$= f \star g \,\|_{x^+=0}^a - (\partial^-)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^- f) \star (\Lambda^{-1/2}\tau^{1/2}g)\Big|_{x^+=0}^a,$$

$$(\partial^3)_{\mathrm{R}}^{-1} f \star \partial_{\mathrm{R}}^3 g\big|_{x^3=0}^a \tag{44}$$

$$= f \star g \,\|_{x^3=0}^a - (\partial^3)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^3 f) \star (\Lambda^{-1/2}g)\Big|_{x^3=0}^a$$

$$+ \lambda\lambda_+(\partial^3)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^- f) \star (\Lambda^{-1/2}\tau^{1/2}L^+ g)\Big|_{x^3=0}^a,$$

$$(\partial^+)_{\mathrm{R}}^{-1} f \star \partial_{\mathrm{R}}^+ g\big|_{x^-=0}^a \tag{45}$$

$$= f \star g \,\|_{x^-=0}^a - (\partial^+)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^+ f) \star (\Lambda^{-1/2}\tau^{-1/2}g)\Big|_{x^-=0}^a$$

$$+ q^{-1}\lambda\lambda_+(\partial^+)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^3 f) \star (\Lambda^{-1/2}L^+ g)\Big|_{x^-=0}^a$$

$$- \lambda^2\lambda_+(\partial^+)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^- f) \star (\Lambda^{-1/2}\tau^{1/2}(L^+)^2 g)\Big|_{x^-=0}^a$$

and

$$(\hat{\partial}^+)_{\mathrm{R}}^{-1} f \star \hat{\partial}_{\mathrm{R}}^+ g\Big|_{x^-=0}^a \tag{46}$$

$$= f \star g \,\|_{x^-=0}^a - (\hat{\partial}^+)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^+ f) \star (\Lambda^{1/2}\tau^{1/2}g)\Big|_{x^-=0}^a,$$

$$(\hat{\partial}^3)_{\mathrm{R}}^{-1} f \star \hat{\partial}_{\mathrm{R}}^3 g\Big|_{x^3=0}^a \tag{47}$$

$$= f \star g \,\|_{x^3=0}^a - (\hat{\partial}^3)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^3 f) \star (\Lambda^{1/2}g)\Big|_{x^3=0}^a$$

$$+ \lambda\lambda_+(\hat{\partial}^3)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^+ f) \star (\Lambda^{1/2}\tau^{1/2}L^- g)\Big|_{x^3=0}^a,$$

$$(\hat{\partial}^-)_{\mathrm{R}}^{-1} f \star \hat{\partial}_{\mathrm{R}}^- g\Big|_{x^+=0}^a \tag{48}$$

$$= f \star g \,\|_{x^+=0}^a - (\hat{\partial}^-)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^- f) \star (\Lambda^{1/2}\tau^{-1/2}g)\Big|_{x^+=0}^a$$

$$+ q\lambda\lambda_+(\hat{\partial}^-)_{\mathrm{R}}^{-1} (\hat{\partial}_{\mathrm{R}}^3 f) \star (\Lambda^{1/2}L^- g)\Big|_{x^+=0}^a$$

$$- \lambda^2\lambda_+(\hat{\partial}^-)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^+ f) \star \left(\Lambda^{1/2}\tau^{1/2}(L^-)^2 g\right)\Big|_{x^+=0}^a.$$

We want to close this section by discussing a method for integration over the entire space. For this purpose we

could define

$$\int_{\mathrm{L}} d_q V\, f \equiv (\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} \triangleright f \qquad (49)$$

for a 3-dimensional volume integral. But this is not the only possibility for defining a global integration, as the operators $(\partial^A)^{-1}$, $A = \pm, 3$, do not mutually commute. However, for functions vanishing at infinity one can show by inserting the representations of (20) that the following identities hold:

$$(\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} \triangleright f \qquad (50)$$
$$= q^{-4}(\partial^-)^{-1}(\partial^3)^{-1}(\partial^+)^{-1} \triangleright f + \mathrm{S.T.}$$
$$= q^{-2}(\partial^3)^{-1}(\partial^-)^{-1}(\partial^+)^{-1} \triangleright f + \mathrm{S.T.}$$
$$= q^{-2}(\partial^3)^{-1}(\partial^+)^{-1}(\partial^-)^{-1} \triangleright f + \mathrm{S.T.}$$
$$= q^{-2}(\partial^-)^{-1}(\partial^+)^{-1}(\partial^3)^{-1} \triangleright f + \mathrm{S.T.}$$
$$= q^{-2}(\partial^+)^{-1}(\partial^-)^{-1}(\partial^3)^{-1} \triangleright f + \mathrm{S.T.}$$

where S.T. stands for neglected surface terms. In the same manner we can also find as an explicit formula for calculating volume integrals

$$(\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} \triangleright f \qquad (51)$$
$$= (\partial^+_{(i=0)})^{-1}(\partial^3_{(i=0)})^{-1}(\partial^-_{(i=0)})^{-1} \triangleright f + \mathrm{O.T.}$$
$$= q^{-4}(D_{q^4}^-)^{-1}(D_{q^2}^3)^{-1}(D_{q^4}^+)^{-1} f(q^{-2}x^+, q^{-2}x^3) + \mathrm{S.T.}$$

if again surface terms are dropped. Using the crossing symmetries (22), (24) and (25) it immediately follows that

$$(\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} \triangleright f \overset{\pm \to \mp}{\underset{q \to 1/q}{\longleftrightarrow}} (\hat\partial^-)^{-1}(\hat\partial^3)^{-1}(\hat\partial^+)^{-1} \tilde\triangleright f \qquad (52)$$

and

$$f \,\tilde\triangleleft\, (\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} \overset{+ \leftrightarrow -}{\longleftrightarrow} (\bar\partial^-)^{-1}(\bar\partial^3)^{-1}(\bar\partial^+)^{-1} \tilde\triangleright f, \qquad (53)$$

$$f \,\triangleleft\, (\bar\partial^-)^{-1}(\bar\partial^3)^{-1}(\bar\partial^+)^{-1} \overset{+ \leftrightarrow -}{\longleftrightarrow} (\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} \triangleright f,$$

where $(\bar\partial^A)^{-1} = -q^6(\hat\partial^A)^{-1}$.

From a direct calculation using the representations in [10] and applying them to the last expressions of (51) one can also verify that our volume integrals are invariant under both translations and rotations. This means, in explicit form, that we have

$$\partial^A \triangleright \int_{\mathrm{L}} d_q V\, f = \int_{\mathrm{L}} d_q V\, \partial^A \triangleright f = \varepsilon(\partial^A) \int_{\mathrm{L}} d_q V\, f = 0,$$
$$(54)$$

$$L^{\pm} \triangleright \int_{\mathrm{L}} d_q V\, f = \int_{\mathrm{L}} d_q V\, L^{\pm} \triangleright f = \varepsilon(L^{\pm}) \int_{\mathrm{L}} d_q V\, f = 0,$$

$$\tau \triangleright \int_{\mathrm{L}} d_q V\, f = \int_{\mathrm{L}} d_q V\, \tau \triangleright f = \varepsilon(\tau) \int_{\mathrm{L}} d_q V\, f = \int_{\mathrm{L}} d_q V\, f,$$

where we have used as a shorthand notation

$$\int_{\mathrm{L}} d_q V\, f \equiv (\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} \triangleright f, \qquad (55)$$

$$\int_{\mathrm{R}} d_q V\, f \equiv f \,\tilde\triangleleft\, (\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1},$$

$$\int_{\mathrm{L}} d_q \overline{V}\, f \equiv (\bar\partial^-)^{-1}(\bar\partial^3)^{-1}(\bar\partial^+)^{-1} \tilde\triangleright f,$$

$$\int_{\mathrm{R}} d_q \overline{V}\, f \equiv f \,\triangleleft\, (\bar\partial^-)^{-1}(\bar\partial^3)^{-1}(\bar\partial^+)^{-1}.$$

In the same way one can check invariance under right representations, hence

$$\left(\int_{\mathrm{L}} d_q V\, f\right) \tilde\triangleleft\, \partial^A = \int_{\mathrm{L}} d_q V\, f \,\tilde\triangleleft\, \partial^A = \varepsilon(\partial^A) \int_{\mathrm{L}} d_q V\, f = 0,$$
$$(56)$$

$$\left(\int_{\mathrm{L}} d_q V\, f\right) \triangleleft L^{\pm} = \int_{\mathrm{L}} d_q V\, f \triangleleft L^{\pm} = \varepsilon(L^{\pm}) \int_{\mathrm{L}} d_q V\, f = 0,$$

$$\left(\int_{\mathrm{L}} d_q V\, f\right) \triangleleft \tau = \int_{\mathrm{L}} d_q V\, f \triangleleft \tau = \varepsilon(\tau) \int_{\mathrm{L}} d_q V\, f$$

$$= \int_{\mathrm{L}} d_q V\, f.$$

Because of the crossing symmetries these identities carry over to all of the other volume integrals. However, one has to notice that in the case of conjugated integrals translation invariance now reads

$$\bar\partial^A \triangleright \int_{\mathrm{L}} d_q \overline{V}\, f = \int_{\mathrm{L}} d_q \overline{V}\, \bar\partial^A \triangleright f = \varepsilon(\bar\partial^A) \int_{\mathrm{L}} d_q \overline{V}\, f = 0,$$
$$(57)$$

$$\left(\int_{\mathrm{L}} d_q \overline{V}\, f\right) \triangleleft \bar\partial^A = \int_{\mathrm{L}} d_q \overline{V}\, f \triangleleft \bar\partial^A = \varepsilon(\bar\partial^A) \int_{\mathrm{L}} d_q \overline{V}\, f = 0,$$

$$\bar\partial^A \,\tilde\triangleright\, \int_{\mathrm{R}} d_q \overline{V}\, f = \int_{\mathrm{R}} d_q \overline{V}\, \bar\partial^A \,\tilde\triangleright\, f = \varepsilon(\bar\partial^A) \int_{\mathrm{R}} d_q \overline{V}\, f = 0,$$

$$\left(\int_{\mathrm{R}} d_q \overline{V}\, f\right) \triangleleft \bar\partial^A = \int_{\mathrm{R}} d_q \overline{V}\, f \triangleleft \bar\partial^A = \varepsilon(\bar\partial^A) \int_{\mathrm{R}} d_q \overline{V}\, f = 0.$$

3 q-Deformed Euclidean space in four dimensions

The 4-dimensional Euclidean space can be treated in very much the same way as the 3-dimensional one. Therefore we will restrict ourselves to stating the results only. Again, we start with the commutation relations [15]

$$\partial^1 \partial^2 = q \partial^2 \partial^1, \quad \partial^1 \partial^3 = q \partial^3 \partial^1, \quad \partial^2 \partial^4 = q \partial^4 \partial^2, \qquad (58)$$

$$\partial^3 \partial^4 = q \partial^3 \partial^2, \quad \partial^2 \partial^3 = \partial^3 \partial^2, \quad \partial^4 \partial^1 = \partial^1 \partial^4 + \lambda \partial^2 \partial^3,$$

where $\lambda = q - q^{-1}$ with $q > 1$, and introduce elements $(\partial^i)^{-1}$, $i = 1, \ldots, 4$, by

$$\partial^i(\partial^i)^{-1} = (\partial^i)^{-1}\partial^i = 1, \quad i = 1, \ldots, 4.$$

Now, the additional commutation relations become

$$(\partial^2)^{-1}\partial^1 = q\partial^1(\partial^2)^{-1}, \quad \partial^2(\partial^1)^{-1} = q(\partial^1)^{-1}\partial^2,$$

$$(\partial^3)^{-1}\partial^1 = q\partial^1(\partial^3)^{-1}, \quad \partial^3(\partial^1)^{-1} = q(\partial^1)^{-1}\partial^3,$$
$$(\partial^4)^{-1}\partial^2 = q\partial^2(\partial^4)^{-1}, \quad \partial^4(\partial^2)^{-1} = q(\partial^2)^{-1}\partial^4,$$
$$(\partial^4)^{-1}\partial^3 = q\partial^3(\partial^4)^{-1}, \quad \partial^4(\partial^3)^{-1} = q(\partial^3)^{-1}\partial^4,$$
$$(\partial^4)^{-1}\partial^1 = \partial^1(\partial^4)^{-1} - q^2\lambda\partial^2\partial^3(\partial^4)^{-2}, \quad (59)$$
$$\partial^4(\partial^1)^{-1} = (\partial^1)^{-1}\partial^4 - q^2\lambda(\partial^1)^{-2}\partial^2\partial^3,$$
$$(\partial^3)^{-1}\partial^2 = \partial^2(\partial^3)^{-1}, \quad \partial^3(\partial^2)^{-1} = (\partial^2)^{-1}\partial^3.$$

Furthermore, we obtain identities of the form

$$(\partial^2)^{-1}(\partial^1)^{-1} = q^{-1}(\partial^1)^{-1}(\partial^2)^{-1}, \quad (60)$$
$$(\partial^3)^{-1}(\partial^1)^{-1} = q^{-1}(\partial^1)^{-1}(\partial^3)^{-1},$$
$$(\partial^4)^{-1}(\partial^2)^{-1} = q^{-1}(\partial^2)^{-1}(\partial^4)^{-1},$$
$$(\partial^4)^{-1}(\partial^3)^{-1} = q^{-1}(\partial^3)^{-1}(\partial^4)^{-1},$$
$$(\partial^3)^{-1}(\partial^2)^{-1} = (\partial^2)^{-1}(\partial^3)^{-1},$$
$$(\partial^4)^{-1}(\partial^1)^{-1} = \sum_{i=0}^{\infty}\lambda^i\,[[i]]_{q^{-2}}!\left(\begin{bmatrix}-1\\i\end{bmatrix}_{q^{-2}}\right)^2$$
$$\cdot(\partial^1)^{-(i+1)}(\partial^2)^i(\partial^3)^i(\partial^4)^{-(i+1)}.$$

Next, the commutation relations with the symmetry generators read

$$L_1^+(\partial^1)^{-1} = q^{-1}(\partial^1)^{-1}L_1^+ + q^{-1}(\partial^1)^{-2}\partial^2, \quad (61)$$
$$L_1^+(\partial^2)^{-1} = q(\partial^2)^{-1}L_1^+,$$
$$L_1^+(\partial^3)^{-1} = q^{-1}(\partial^3)^{-1}L_1^+ - q^{-1}(\partial^3)^{-2}\partial^4,$$
$$L_1^+(\partial^4)^{-1} = q(\partial^4)^{-1}L_1^+,$$

$$L_2^+(\partial^1)^{-1} = q^{-1}(\partial^1)^{-1}L_2^+ + q^{-1}(\partial^1)^{-2}\partial^3, \quad (62)$$
$$L_2^+(\partial^2)^{-1} = q^{-1}(\partial^2)^{-1}L_2^+ - q^{-1}(\partial^2)^{-2}\partial^4,$$
$$L_2^+(\partial^3)^{-1} = q(\partial^3)^{-1}L_2^+,$$
$$L_2^+(\partial^4)^{-1} = q(\partial^4)^{-1}L_2^+,$$

$$L_1^-(\partial^1)^{-1} = q^{-1}(\partial^1)^{-1}L_1^-, \quad (63)$$
$$L_1^-(\partial^2)^{-1} = q(\partial^2)^{-1}L_1^- + q^3\partial^1(\partial^2)^{-2},$$
$$L_1^-(\partial^3)^{-1} = q^{-1}(\partial^3)^{-1}L_1^-,$$
$$L_1^-(\partial^4)^{-1} = q(\partial^4)^{-1}L_1^- - q^3\partial^3(\partial^4)^{-2},$$

$$L_2^-(\partial^1)^{-1} = q^{-1}(\partial^1)^{-1}L_2^-, \quad (64)$$
$$L_2^-(\partial^2)^{-1} = q^{-1}(\partial^2)^{-1}L_2^-,$$
$$L_2^-(\partial^3)^{-1} = q(\partial^3)^{-1}L_2^- + q^3\partial^1(\partial^3)^{-2},$$
$$L_2^-(\partial^4)^{-1} = q(\partial^4)^{-1}L_2^- - q^3\partial^2(\partial^4)^{-2},$$

$$K_1(\partial^1)^{-1} = q(\partial^1)^{-1}K_1, \quad (65)$$

$$K_1(\partial^2)^{-1} = q^{-1}(\partial^2)^{-1}K_1,$$
$$K_1(\partial^3)^{-1} = q(\partial^3)^{-1}K_1,$$
$$K_1(\partial^4)^{-1} = q^{-1}(\partial^4)^{-1}K_1,$$

$$K_2(\partial^1)^{-1} = q(\partial^1)^{-1}K_2, \quad (66)$$
$$K_2(\partial^2)^{-1} = q(\partial^2)^{-1}K_2,$$
$$K_2(\partial^3)^{-1} = q^{-1}(\partial^3)^{-1}K_2,$$
$$K_2(\partial^4)^{-1} = q^{-1}(\partial^4)^{-1}K_2.$$

Applying the substitutions

$$\partial^i \to \hat{\partial}^i, \quad (\partial^i)^{-1} \to (\hat{\partial}^i)^{-1}, \quad i = 1,\dots,4, \quad (67)$$

we get the corresponding relations for the second differential calculus spanned by the conjugated derivatives.

With the representations in [10] the same reasonings we have already applied to the 3-dimensional Euclidean space lead immediately to the expressions

$$(\hat{\partial}^1)^{-1} \triangleright f = q\sum_{k=0}^{\infty}(-\lambda)^k q^{-k(k+1)}\left[x^1(D_{q^{-2}}^4)^{-1}D_{q^{-2}}^2 D_{q^{-2}}^3\right]^k$$
$$\cdot(D_{q^{-2}}^4)^{-1}f(q^{k+1}x^2, q^{k+1}x^3), \quad (68)$$
$$(\hat{\partial}^2)^{-1} \triangleright f = (D_{q^{-2}}^3)^{-1}f(qx^1),$$
$$(\hat{\partial}^3)^{-1} \triangleright f = (D_{q^{-2}}^2)^{-1}f(qx^1),$$
$$(\hat{\partial}^4)^{-1} \triangleright f = q^{-1}(D_{q^{-2}}^1)^{-1}f.$$

In complete analogy to the 3-dimensional case we have the transformation

$$(\partial^i)^{-1} \tilde{\triangleright} f \overset{i\,\to\,i'}{\underset{q\,\leftrightarrow\,1/q}{\longleftrightarrow}} (\hat{\partial}^{i'})^{-1} \triangleright f, \quad i' = 5 - i, \quad (69)$$

whose explicit form is given by the substitutions

$$x^{\pm} \to x^{\mp}, \quad q^{\pm 1} \to q^{\mp 1}, \quad \hat{n}^i \to -\hat{n}^{i'}, \quad (70)$$
$$D_{q^a}^i \to D_{q^{-a}}^{i'}, \quad (D_{q^a}^i)^{-1} \to (D_{q^{-a}}^{i'})^{-1}.$$

For the relationship between right and left integrals we now have

$$f \triangleleft (\partial^i)^{-1} \overset{j\leftrightarrow j'}{\longleftrightarrow} -q^4(\hat{\partial}^{i'})^{-1} \triangleright f, \quad (71)$$
$$f \tilde{\triangleleft} (\hat{\partial}^i)^{-1} \overset{j\leftrightarrow j'}{\longleftrightarrow} -q^{-4}(\partial^{i'})^{-1} \tilde{\triangleright} f,$$

which symbolizes that we have to apply the substitutions

$$x^j \to x^{j'}, \quad D_{q^a}^j \to D_{q^a}^{j'}, \quad (D_{q^a}^j)^{-1} \to (D_{q^a}^{j'})^{-1},$$
$$\hat{n}^j \to \hat{n}^{j'}. \quad (72)$$

Next, we would like to present the formulae for integration by parts. For left integrals belonging to the differential calculus with unhatted derivatives we find

$$(\partial^1)_{\mathrm{L}}^{-1}(\partial_{\mathrm{L}}^1 f) \star g\big|_{x^4=0}^a \quad (73)$$

$$= f \star g \parallel^a_{x^4=0} - (\partial^1)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{1/2}K_2^{1/2}f) \star \partial^1_{\mathrm{L}}g \Big|^a_{x^4=0},$$

$$(\partial^2)^{-1}_{\mathrm{L}}(\partial^2_{\mathrm{L}}f) \star g \Big|^a_{x^3=0} \tag{74}$$

$$= f \star g \parallel^a_{x^3=0} - (\partial^2)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{-1/2}K_2^{1/2}f) \star \partial^2_{\mathrm{L}}g \Big|^a_{x^3=0}$$

$$- q\lambda \, (\partial^2)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{1/2}K_2^{1/2}L_1^+f) \star \partial^1_{\mathrm{L}}g \Big|^a_{x^3=0},$$

$$(\partial^3)^{-1}_{\mathrm{L}}(\partial^3_{\mathrm{L}}f) \star g \Big|^a_{x^2=0} \tag{75}$$

$$= f \star g \parallel^a_{x^2=0} - (\partial^3)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{1/2}K_2^{-1/2}f) \star \partial^3_{\mathrm{L}}g \Big|^a_{x^3=0}$$

$$- q\lambda \, (\partial^3)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{1/2}K_2^{1/2}L_2^+f) \star \partial^1_{\mathrm{L}}g \Big|^a_{x^2=0},$$

$$(\partial^4)^{-1}_{\mathrm{L}}(\partial^4_{\mathrm{L}}f) \star g \Big|^a_{x^1=0} \tag{76}$$

$$= f \star g \parallel^a_{x^1=0} - (\partial^4)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{-1/2}K_2^{-1/2}f) \star \partial^4_{\mathrm{L}}g \Big|^a_{x^1=0}$$

$$+ q\lambda \, (\partial^4)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{1/2}K_2^{-1/2}L_1^+f) \star \partial^3_{\mathrm{L}}g \Big|^a_{x^1=0}$$

$$+ q\lambda \, (\partial^4)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{-1/2}K_2^{1/2}L_2^+f) \star \partial^2_{\mathrm{L}}g \Big|^a_{x^1=0}$$

$$+ q^2\lambda^2 \, (\partial^4)^{-1}_{\mathrm{L}}(\Lambda^{1/2}K_1^{1/2}K_2^{1/2}L_1^+L_2^+f) \star \partial^1_{\mathrm{L}}g \Big|^a_{x^1=0}.$$

For the second differential calculus with hatted derivatives one can compute likewise

$$(\hat{\partial}^1)^{-1}_{\mathrm{L}}(\hat{\partial}^1_{\mathrm{L}}f) \star g \Big|^a_{x^4=0} \tag{77}$$

$$= f \star g \parallel^a_{x^4=0} - (\hat{\partial}^1)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{-1/2}K_2^{-1/2}f) \star \hat{\partial}^1_{\mathrm{L}}g \Big|^a_{x^4=0}$$

$$+ q^{-1}\lambda \, (\hat{\partial}^1)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{1/2}K_2^{-1/2}L_1^-f) \star \hat{\partial}^2_{\mathrm{L}}g \Big|^a_{x^4=0}$$

$$+ q^{-1}\lambda \, (\hat{\partial}^1)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{-1/2}K_2^{1/2}L_2^-f) \star \hat{\partial}^3_{\mathrm{L}}g \Big|^a_{x^4=0}$$

$$+ q^{-2}\lambda^2 \, (\hat{\partial}^1)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{1/2}K_2^{-1/2}L_1^-L_2^-f) \star \hat{\partial}^4_{\mathrm{L}}g \Big|^a_{x^4=0},$$

$$(\hat{\partial}^2)^{-1}_{\mathrm{L}}(\hat{\partial}^2_{\mathrm{L}}f) \star g \Big|^a_{x^3=0} \tag{78}$$

$$= f \star g \parallel^a_{x^3=0} - (\hat{\partial}^2)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{1/2}K_2^{-1/2}f) \star \hat{\partial}^2_{\mathrm{L}}g \Big|^a_{x^3=0}$$

$$- q^{-1}\lambda \, (\hat{\partial}^2)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{1/2}K_2^{1/2}L_2^-f) \star \hat{\partial}^4_{\mathrm{L}}g \Big|^a_{x^3=0},$$

$$(\hat{\partial}^3)^{-1}_{\mathrm{L}}(\hat{\partial}^3_{\mathrm{L}}f) \star g \Big|^a_{x^2=0} \tag{79}$$

$$= f \star g \parallel^a_{x^2=0} - (\hat{\partial}^3)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{-1/2}K_2^{1/2}f) \star \hat{\partial}^3_{\mathrm{L}}g \Big|^a_{x^2=0}$$

$$- q^{-1}\lambda \, (\hat{\partial}^3)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{1/2}K_2^{1/2}L_1^-f) \star \hat{\partial}^4_{\mathrm{L}}g \Big|^a_{x^2=0},$$

$$(\hat{\partial}^4)^{-1}_{\mathrm{L}}(\hat{\partial}^4_{\mathrm{L}}f) \star g \Big|^a_{x^1=0} \tag{80}$$

$$= f \star g \parallel^a_{x^1=0} - (\hat{\partial}^4)^{-1}_{\mathrm{L}}(\Lambda^{-1/2}K_1^{1/2}K_2^{1/2}f) \star \hat{\partial}^4_{\mathrm{L}}g \Big|^a_{x^1=0}.$$

Now we come to the corresponding formulae for right integrals which in the case of the differential calculus with unhatted derivatives read

$$(\partial^1)^{-1}_{\mathrm{R}}f \star \partial^1_{\mathrm{R}}g \Big|^a_{x^4=0} \tag{81}$$

$$= f \star g \parallel^a_{x^4=0}$$

$$- (\partial^1)^{-1}_{\mathrm{R}}(\partial^1_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{-1/2}K_2^{-1/2}g) \Big|^a_{x^4=0},$$

$$(\partial^2)^{-1}_{\mathrm{R}}f \star \partial^2_{\mathrm{R}}g \Big|^a_{x^3=0} \tag{82}$$

$$= f \star g \parallel^a_{x^3=0}$$

$$- (\partial^2)^{-1}_{\mathrm{R}}(\partial^2_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{1/2}K_2^{-1/2}g) \Big|^a_{x^3=0}$$

$$+ \lambda \, (\partial^2)^{-1}_{\mathrm{R}}(\partial^1_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{1/2}K_2^{-1/2}L_1^+g) \Big|^a_{x^3=0},$$

$$(\partial^3)^{-1}_{\mathrm{R}}f \star \partial^3_{\mathrm{R}}g \Big|^a_{x^2=0} \tag{83}$$

$$= f \star g \parallel^a_{x^2=0}$$

$$- (\partial^3)^{-1}_{\mathrm{R}}(\partial^3_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{-1/2}K_2^{1/2}g) \Big|^a_{x^2=0}$$

$$+ \lambda \, (\partial^3)^{-1}_{\mathrm{R}}(\partial^1_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{-1/2}K_2^{1/2}L_2^+g) \Big|^a_{x^2=0},$$

$$(\partial^4)^{-1}_{\mathrm{R}}f \star \partial^4_{\mathrm{R}}g \Big|^a_{x^1=0} \tag{84}$$

$$= f \star g \parallel^a_{x^1=0} - (\partial^4)^{-1}_{\mathrm{R}}(\partial^4_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{1/2}K_2^{1/2}g) \Big|^a_{x^1=0}$$

$$- \lambda \, (\partial^4)^{-1}_{\mathrm{R}}(\partial^3_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{1/2}K_2^{1/2}L_1^+g) \Big|^a_{x^1=0}$$

$$- \lambda \, (\partial^4)^{-1}_{\mathrm{R}}(\partial^2_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{1/2}K_2^{1/2}L_2^+g) \Big|^a_{x^1=0}$$

$$+ \lambda^2 \, (\partial^4)^{-1}_{\mathrm{R}}(\partial^1_{\mathrm{R}}f) \star (\Lambda^{-1/2}K_1^{1/2}K_2^{1/2}L_1^+L_2^+g) \Big|^a_{x^1=0}.$$

In the same manner we get for the second differential calculus with unhatted derivatives

$$(\hat{\partial}^1)^{-1}_{\mathrm{R}}f \star \hat{\partial}^1_{\mathrm{R}}g \Big|^a_{x^4=0} \tag{85}$$

$$= f \star g \parallel^a_{x^4=0}$$

$$- (\hat{\partial}^1)^{-1}_{\mathrm{R}}(\hat{\partial}^1_{\mathrm{R}}f) \star (\Lambda^{1/2}K_1^{1/2}K_2^{1/2}g) \Big|^a_{x^4=0}$$

$$- \lambda \, (\hat{\partial}^1)^{-1}_{\mathrm{R}}(\hat{\partial}^2_{\mathrm{R}}f) \star (\Lambda^{1/2}K_1^{1/2}K_2^{1/2}L_1^-g) \Big|^a_{x^4=0}$$

$$- \lambda \, (\hat{\partial}^1)^{-1}_{\mathrm{R}}(\hat{\partial}^3_{\mathrm{R}}f) \star (\Lambda^{1/2}K_1^{1/2}K_2^{1/2}L_2^-g) \Big|^a_{x^4=0}$$

$$+ \lambda^2 \, (\hat{\partial}^1)^{-1}_{\mathrm{R}}(\hat{\partial}^4_{\mathrm{R}}f) \star (\Lambda^{1/2}K_1^{1/2}K_2^{1/2}L_1^-L_2^-g) \Big|^a_{x^4=0},$$

$$(\hat{\partial}^2)_{\mathrm{R}}^{-1} f \star \hat{\partial}_{\mathrm{R}}^2 g \Big|_{x^3=0}^{a} \tag{86}$$

$$= f \star g \, \|_{x^3=0}^{a}$$

$$- (\hat{\partial}^2)_{\mathrm{R}}^{-1} (\hat{\partial}_{\mathrm{R}}^2 f) \star (\Lambda^{1/2} K_1^{-1/2} K_2^{1/2} g) \Big|_{x^3=0}^{a}$$

$$+ \lambda \, (\hat{\partial}^2)_{\mathrm{R}}^{-1} (\hat{\partial}_{\mathrm{R}}^4 f) \star (\Lambda^{1/2} K_1^{-1/2} K_2^{1/2} L_2^- g) \Big|_{x^3=0}^{a},$$

$$(\hat{\partial}^3)_{\mathrm{R}}^{-1} f \star \hat{\partial}_{\mathrm{R}}^3 g \Big|_{x^2=0}^{a} \tag{87}$$

$$= f \star g \, \|_{x^2=0}^{a} - (\hat{\partial}^3)_{\mathrm{R}}^{-1} (\hat{\partial}_{\mathrm{R}}^3 f) \star (\Lambda^{1/2} K_1^{-1/2} K_2^{-1/2} g) \Big|_{x^2=0}^{a}$$

$$+ \lambda \, (\hat{\partial}^3)_{\mathrm{R}}^{-1} (\hat{\partial}_{\mathrm{R}}^4 f) \star (\Lambda^{1/2} K_1^{1/2} K_2^{-1/2} L_1^- g) \Big|_{x^2=0}^{a},$$

$$(\hat{\partial}^4)_{\mathrm{R}}^{-1} f \star \hat{\partial}_{\mathrm{R}}^4 g \Big|_{x^1=0}^{a} \tag{88}$$

$$= f \star g \, \|_{x^1=0}^{a}$$

$$- (\hat{\partial}^4)_{\mathrm{R}}^{-1} (\hat{\partial}_{\mathrm{R}}^4 f) \star (\Lambda^{1/2} K_1^{-1/2} K_2^{-1/2} g) \Big|_{x^1=0}^{a}.$$

Finally, we would like to present the relations concerning our method for integration over the whole space. If surface terms are neglected, we can again state the identities

$$(\partial^1)^{-1}(\partial^2)^{-1}(\partial^3)^{-1}(\partial^4)^{-1}$$

$$= (\partial^1)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^4)^{-1} \tag{89}$$

$$= q(\partial^1)^{-1}(\partial^2)^{-1}(\partial^4)^{-1}(\partial^3)^{-1}$$

$$= q(\partial^1)^{-1}(\partial^3)^{-1}(\partial^4)^{-1}(\partial^2)^{-1}$$

$$= q(\partial^2)^{-1}(\partial^1)^{-1}(\partial^3)^{-1}(\partial^4)^{-1}$$

$$= q(\partial^3)^{-1}(\partial^1)^{-1}(\partial^2)^{-1}(\partial^4)^{-1}$$

$$= q^3(\partial^2)^{-1}(\partial^4)^{-1}(\partial^3)^{-1}(\partial^1)^{-1}$$

$$= q^3(\partial^3)^{-1}(\partial^4)^{-1}(\partial^2)^{-1}(\partial^1)^{-1}$$

$$= q^3(\partial^4)^{-1}(\partial^2)^{-1}(\partial^1)^{-1}(\partial^3)^{-1}$$

$$= q^3(\partial^4)^{-1}(\partial^3)^{-1}(\partial^1)^{-1}(\partial^2)^{-1}$$

$$= q^4(\partial^4)^{-1}(\partial^2)^{-1}(\partial^3)^{-1}(\partial^1)^{-1}$$

$$= q^4(\partial^4)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^1)^{-1}$$

$$= q^2 \; \times \text{ remaining combinations.}$$

Let us note that the quantities $(\hat{\partial}^i)^{-1}$, $i = 1, \ldots, 4$, obey the same relations. Since the results of the various volume integrals in (89) differ by normalization factors only, we can restrict attention to one of the above expressions. Thus, for performing the integration over the whole space it is sufficient to consider the following formula:

$$(\hat{\partial}^1)^{-1}(\hat{\partial}^2)^{-1}(\hat{\partial}^3)^{-1}(\hat{\partial}^4)^{-1} \triangleright f \tag{90}$$

$$= q^4 (D_{q^{-2}}^4)^{-1} (D_{q^{-2}}^3)^{-1} (D_{q^{-2}}^2)^{-1} (D_{q^{-2}}^1)^{-1}$$

$$\cdot f(q^2 x^1, q x^2, q x^3).$$

It is quite clear that the transformations (69) and (71) carry over into our formulae for volume integrals. Hence,

$$(\hat{\partial}^1)^{-1}(\hat{\partial}^2)^{-1}(\hat{\partial}^3)^{-1}(\hat{\partial}^4)^{-1} \triangleright f \tag{91}$$

$$\overset{\substack{i \,\to\, i' \\ q \,\rightleftarrows\, 1/q}}{\longleftrightarrow} (\partial^4)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^1)^{-1} \tilde{\triangleright} f$$

and

$$f \,\tilde{\triangleleft}\, (\bar{\partial}^1)^{-1}(\bar{\partial}^2)^{-1}(\bar{\partial}^3)^{-1}(\bar{\partial}^4)^{-1} \tag{92}$$

$$\overset{j \leftrightarrow j'}{\longleftrightarrow} (\partial^4)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^1)^{-1} \tilde{\triangleright} f,$$

$$f \,\triangleleft\, (\partial^4)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^1)^{-1} \tag{93}$$

$$\overset{j \leftrightarrow j'}{\longleftrightarrow} (\bar{\partial}^1)^{-1}(\bar{\partial}^2)^{-1}(\bar{\partial}^3)^{-1}(\bar{\partial}^4)^{-1} \triangleright f,$$

where $(\partial^i)^{-1} = -q^4 (\hat{\partial}^i)^{-1}$. With the same reasoning applied to q-deformed Euclidean space in three dimensions we can immediately verify rotation and translation invariance of our 4-dimensional volume integrals. Using the notation

$$\int_{\mathrm{L}} d_q V f \equiv (\partial^4)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^1)^{-1} \tilde{\triangleright} f, \tag{94}$$

$$\int_{\mathrm{R}} d_q V f \equiv f \triangleleft (\partial^4)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^1)^{-1},$$

$$\int_{\mathrm{L}} d_q \overline{V} f \equiv (\bar{\partial}^1)^{-1}(\bar{\partial}^2)^{-1}(\bar{\partial}^3)^{-1}(\bar{\partial}^4)^{-1} \triangleright f,$$

$$\int_{\mathrm{R}} d_q \overline{V} f \equiv f \,\tilde{\triangleleft}\, (\bar{\partial}^1)^{-1}(\bar{\partial}^2)^{-1}(\bar{\partial}^3)^{-1}(\bar{\partial}^4)^{-1},$$

we can explicitly write

$$\partial^i \tilde{\triangleright} \int_{\mathrm{L/R}} d_q V f = \int_{\mathrm{L/R}} d_q V \partial^i \tilde{\triangleright} f \tag{95}$$

$$= \varepsilon(\partial^i) \int_{\mathrm{L/R}} d_q V f = 0,$$

$$L_j^{\pm} \triangleright \int_{\mathrm{L/R}} d_q V f = \int_{\mathrm{L/R}} d_q V L_j^{\pm} \triangleright f$$

$$= \varepsilon(L_j^{\pm}) \int_{\mathrm{L/R}} d_q V f = 0,$$

$$K_j \triangleright \int_{\mathrm{L/R}} d_q V f = \int_{\mathrm{L/R}} d_q V K_j \triangleright f$$

$$= \varepsilon(K_j) \int_{\mathrm{L/R}} d_q V f = \int_{\mathrm{L/R}} d_q V f,$$

$$\bar{\partial}^i \triangleright \int_{\mathrm{L/R}} d_q \overline{V} f = \int_{\mathrm{L/R}} d_q \overline{V} \bar{\partial}^i \triangleright f \tag{96}$$

$$= \varepsilon(\bar{\partial}^i) \int_{\mathrm{L/R}} d_q \overline{V} f = 0,$$

$$L_j^\pm \triangleright \int_{\mathrm{L/R}} d_q \overline{V} f = \int_{\mathrm{L/R}} d_q \overline{V} L_j^\pm \triangleright f$$

$$= \varepsilon(L_j^\pm) \int_{\mathrm{L/R}} d_q \overline{V} f = 0,$$

$$K_j \triangleright \int_{\mathrm{L/R}} d_q \overline{V} f = \int_{\mathrm{L/R}} d_q \overline{V} K_j \triangleright f$$

$$= \varepsilon(K_j) \int_{\mathrm{L/R}} d_q \overline{V} f = \int_{\mathrm{L/R}} d_q \overline{V} f$$

and

$$\left(\int_{\mathrm{L/R}} d_q V f \right) \tilde{\triangleleft} \, \partial^i = \int_{\mathrm{L/R}} d_q V f \, \tilde{\triangleleft} \, \partial^i \tag{97}$$

$$= \varepsilon(\partial^i) \int_{\mathrm{L/R}} d_q V f = 0,$$

$$\left(\int_{\mathrm{L/R}} d_q V f \right) \triangleleft L_j^\pm = \int_{\mathrm{L/R}} d_q V f \triangleleft L_j^\pm$$

$$= \varepsilon(L_j^\pm) \int_{\mathrm{L/R}} d_q V f = 0,$$

$$\left(\int_{\mathrm{L/R}} d_q V f \right) \triangleleft K_j = \int_{\mathrm{L/R}} d_q V f \triangleleft K_j$$

$$= \varepsilon(K_j) \int_{\mathrm{L/R}} d_q V f = \int_{\mathrm{L/R}} d_q V f,$$

$$\left(\int_{\mathrm{L/R}} d_q \overline{V} f \right) \triangleleft \bar{\partial}^i = \int_{\mathrm{L/R}} d_q \overline{V} f \triangleleft \bar{\partial}^i \tag{98}$$

$$= \varepsilon(\bar{\partial}^i) \int_{\mathrm{L/R}} d_q V f = 0,$$

$$\left(\int_{\mathrm{L/R}} d_q \overline{V} f \right) \triangleleft L_j^\pm = \int_{\mathrm{L/R}} d_q \overline{V} f \triangleleft L_j^\pm$$

$$= \varepsilon(L_j^\pm) \int_{\mathrm{L/R}} d_q \overline{V} f = 0,$$

$$\left(\int_{\mathrm{L/R}} d_q \overline{V} f \right) \triangleleft K_j = \int_{\mathrm{L/R}} d_q \overline{V} f \triangleleft K_j$$

$$= \varepsilon(K_j) \int_{\mathrm{L/R}} d_q \overline{V} f = \int_{\mathrm{L/R}} d_q \overline{V} f.$$

4 q-Deformed Minkowski-space

In principle all considerations of the previous two sections pertain equally to q-deformed Minkowski space [12, 16–18][2], apart from the fact that the results now entail a more

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[2] For a different version of q-deformed Minkowski space see also [19]

involved structure. The partial derivatives of q-deformed Minkowski space satisfy the relations

$$\partial^\mu \partial^0 = \partial^0 \partial^\mu, \quad \mu \in \{0, \pm, 3/0\}, \tag{99}$$

$$\partial^- \partial^{3/0} = q^2 \partial^{3/0} \partial^-,$$

$$\partial^+ \partial^{3/0} = q^{-2} \partial^{3/0} \partial^+,$$

$$\partial^- \partial^+ - \partial^+ \partial^- = \lambda(\partial^{3/0}\partial^{3/0} + \partial^0\partial^{3/0}),$$

with $\lambda = q - q^{-1}$ and $q > 1$. As usual, we introduce inverse elements $(\partial^\mu)^{-1}$, $\mu \in \{0, \pm, 3/0\}$, by

$$(\partial^\mu)^{-1} \partial^\mu = \partial^\mu (\partial^\mu)^{-1} = 1, \quad \mu \in \{0, \pm, 3/0\}. \tag{100}$$

From these requirements we get the relations

$$\partial^0 (\partial^\mu)^{-1} = (\partial^\mu)^{-1} \partial^0, \quad \mu \in \{0, \pm, 3/0\}, \tag{101}$$

$$(\partial^{3/0})^{-1} \partial^\pm = q^{\mp 2} \partial^\pm (\partial^{3/0})^{-1},$$

$$(\partial^\pm)^{-1} \partial^{3/0} = q^{\pm 2} \partial^{3/0} (\partial^\pm)^{-1},$$

$$\partial^- (\partial^+)^{-1}$$

$$= (\partial^+)^{-1} \partial^- - q^{-2}\lambda(\partial^+)^{-2}(q^{-2}\partial^{3/0} + \partial^0)\partial^{3/0},$$

$$(\partial^-)^{-1} \partial^+$$

$$= \partial^+ (\partial^-)^{-1} - q^{-2}\lambda\partial^{3/0}(q^{-2}\partial^{3/0} + \partial^0)(\partial^-)^{-2}.$$

Furthermore, we have

$$(\partial^0)^{-1}(\partial^\mu)^{-1} = (\partial^\mu)^{-1}(\partial^0)^{-1}, \quad \mu \in \{0, \pm, 3/0\},$$

$$(\partial^{3/0})^{-1}(\partial^\pm)^{-1} = q^{\pm 2}(\partial^+)^{-1}(\partial^{3/0})^{-1}, \tag{102}$$

$$(\partial^-)^{-1}(\partial^+)^{-1} = q^2 \sum_{i=0}^\infty (\lambda \lambda_+^{-1})^i \, [[i]]_{q^2}! \left(\begin{bmatrix} -1 \\ i \end{bmatrix}_{q^{-2}} \right)^2$$

$$\cdot \sum_{j+k=i} (-q^6)^k q^{i(i+2k)} \begin{bmatrix} i \\ k \end{bmatrix}_{q^2} \sum_{p=0}^k (q^{4j}\lambda_+)^p$$

$$\cdot (\partial^+)^{p-(i+1)}(\partial^{3/0})^{2j}(S_q)_{k,p}(\partial^0, \partial^{3/0})(\partial^-)^{p-(i+1)},$$

where $S_{k,p}$ is a polynomial of degree $2(k - p)$ with its explicit form presented in Appendix A.

Next, we come to the commutation relations involving the Lorentz generators [17,20,21]. Explicitly, they are given by

$$T^+(\partial^0)^{-1} = (\partial^0)^{-1}T^+, \tag{103}$$

$$T^+(\partial^{3/0})^{-1} = (\partial^{3/0})^{-1}T^+ - q^{1/2}\lambda_+^{1/2}(\partial^{3/0})^{-2}\partial^+,$$

$$T^+(\partial^+)^{-1} = q^2(\partial^+)^{-1}T^+,$$

$$T^+(\partial^-)^{-1} = q^{-2}(\partial^-)^{-1}T^+$$

$$\qquad - q^{-1/2}\lambda_+^{1/2}(\partial^-)^{-2}(\partial^{3/0} + q^{-2}\partial^0),$$

$$T^-(\partial^0)^{-1} = (\partial^0)^{-1}T^- \tag{104}$$

$$T^-(\partial^{3/0})^{-1} = (\partial^{3/0})^{-1}T^- - q^{-1/2}\lambda_+^{1/2}(\partial^{3/0})^{-2}\partial^-,$$

$$T^-(\partial^-)^{-1} = q^{-2}(\partial^-)^{-1}T^-,$$

$$T^-(\partial^+)^{-1} = q^2(\partial^+)^{-1}T^-$$
$$-q^{1/2}\lambda_+^{1/2}(\partial^+)^{-2}(\partial^{3/0} + q^2\partial^0),$$

$$\tau^3(\partial^0)^{-1} = (\partial^0)^{-1}\tau^3, \tag{105}$$

$$\tau^3(\partial^{3/0})^{-1} = (\partial^{3/0})^{-1}\tau^3,$$

$$\tau^3(\partial^+)^{-1} = q^4(\partial^+)^{-1}\tau^3,$$

$$\tau^3(\partial^-)^{-1} = q^{-4}(\partial^-)^{-1}\tau^3,$$

$$T^2(\partial^{3/0})^{-1} = q(\partial^{3/0})^{-1}T^2, \tag{106}$$

$$T^2(\partial^+)^{-1} = q^{-1}(\partial^+)^{-1}T^2,$$

$$T^2(\partial^-)^{-1} = q(\partial^-)^{-1}T^2 - q^{-3/2}\lambda_+^{-1/2}\partial^{3/0}(\partial^-)^{-2}\tau^1,$$

$$T^2(\partial^3)^{-1} = (T^2 \triangleright (\partial^3)^{-1})\tau^1 + (\sigma^2 \triangleright (\partial^3)^{-1})T^2,$$

$$S^1(\partial^{3/0})^{-1} = q^{-1}(\partial^{3/0})^{-1}S^1, \tag{107}$$

$$S^1(\partial^-)^{-1} = q^{-1}(\partial^-)^{-1}S^1,$$

$$S^1(\partial^+)^{-1} = q(\partial^+)^{-1}S^1 + q^{-1/2}\lambda_+^{-1/2}(\partial^+)^{-2}\partial^{3/0}\sigma^2,$$

$$S^1(\partial^3)^{-1} = (S^1 \triangleright (\partial^3)^{-1})\sigma^2 + (\tau^1 \triangleright (\partial^3)^{-1})S^1,$$

$$\tau^1(\partial^{3/0})^{-1} = q^{-1}(\partial^{3/0})^{-1}\tau^1, \tag{108}$$

$$\tau^1(\partial^-)^{-1} = q(\partial^-)^{-1}\tau^1,$$

$$\tau^1(\partial^+)^{-1} = q^{-1}(\partial^+)^{-1}\tau^1$$
$$+q^{-1/2}\lambda_+^{-1/2}\lambda^2(\partial^+)^{-2}\partial^{3/0}T^2,$$

$$\tau^1(\partial^3)^{-1} = (\tau^1 \triangleright (\partial^3)^{-1})\tau^1 + \lambda^2(S^1 \triangleright (\partial^3)^{-1})T^2,$$

$$\sigma^2(\partial^{3/0})^{-1} = q(\partial^{3/0})^{-1}\sigma^2, \tag{109}$$

$$\sigma^2(\partial^+)^{-1} = q(\partial^+)^{-1}\sigma^2,$$

$$\sigma^2(\partial^-)^{-1} = q^{-1}(\partial^-)^{-1}\sigma^2$$
$$-q^{1/2}\lambda_+^{-1/2}\lambda^2(\partial^-)^{-2}\partial^{3/0}S^1,$$

$$\sigma^2(\partial^3)^{-1} = (\sigma^2 \triangleright (\partial^3)^{-1})\sigma^2 + \lambda^2(T^2 \triangleright (\partial^3)^{-1})S^1,$$

where we have to insert the actions

$$T^2 \triangleright (\partial^3)^{-1} = q^{-3/2}\sum_{k=0}^{\infty}(q^2\alpha_0)^k (K_{-1})_{1,q^2}^{(k,k+1)} \tag{110}$$

$$\cdot \sum_{0 \leq i+j \leq k}\lambda_+^{j-1/2}\binom{k}{i}(\partial^+)^{j+1}(a_q(q^{2j}\partial^{3/0}))^i$$

$$\cdot (S_q)_{k-i,j}(\partial^0, \partial^{3/0})(\partial^0 + 2q^{2j+1}\lambda_+^{-1}\partial^{3/0})^{-2(k+1)}(\partial^-)^j,$$

$$S^1 \triangleright (\partial^3)^{-1} = -q^{-3/2}\sum_{k=0}^{\infty}(q^{-2}\alpha_0)^k (K_{-1})_{1,q^{-2}}^{(k,k+1)} \tag{111}$$

$$\cdot \sum_{0 \leq i+j \leq k}\lambda_+^{j-1/2}\binom{k}{i}(\partial^+)^j(a_q(q^{2j}\partial^{3/0}))^i$$

$$\cdot (S_q)_{k-i,j}(\partial^0, \partial^{3/0})(\partial^0 + 2q^{2j+1}\lambda_+^{-1}\partial^{3/0})^{-2(k+1)}(\partial^-)^{j+1},$$

$$\tau^1 \triangleright (\partial^3)^{-1} = q\sum_{k=0}^{\infty}(q^2\alpha_0)^k (K_{-1})_{1,q^2}^{(k,k)} \tag{112}$$

$$\cdot \sum_{0 \leq i+j \leq k}\lambda_+^{j}\binom{k}{i}(\partial^+)^j(a_q(q^{2j}\partial^{3/0}))^i$$

$$\cdot (S_q)_{k-i,j}(\partial^0, \partial^{3/0})(\partial^0 + 2q^{2j+1}\lambda_+^{-1}\partial^{3/0})^{-2k-1}(\partial^-)^j,$$

$$\sigma^2 \triangleright (\partial^3)^{-1} = q^{-1}\sum_{k=0}^{\infty}(q^{-2}\alpha_0)^k (K_{-1})_{1,q^{-2}}^{(k,k)} \tag{113}$$

$$\cdot \sum_{0 \leq i+j \leq k}\lambda_+^{j}\binom{k}{i}(\partial^+)^j(a_{q^{-1}}(q^{2j}\partial^{3/0}))^i$$

$$\cdot (S_q)_{k-i,j}(\partial^0, \partial^{3/0})(\partial^0 + 2q^{2j-1}\lambda_+^{-1}\partial^{3/0})^{-2k-1}(\partial^-)^j$$

with

$$\alpha_0 = -\frac{\lambda^2}{\lambda_+^2}. \tag{114}$$

Note that these expressions have been formulated by using the abbreviations and conventions listed in Appendix A. Applying the substitutions

$$\partial^\mu \to \hat{\partial}^\mu, \qquad (\partial^\mu)^{-1} \to (\hat{\partial}^\mu)^{-1}, \quad \mu \in \{0, \pm, 3/0\}, \tag{115}$$

to the formulae (99)–(113) yields the corresponding identities for the differential calculus of the hatted derivatives.

As in the Euclidean case the representations of the partial derivatives of q-deformed Minkowski space split into two parts [10]. Hence

$$\hat{\partial}^\mu \triangleright F = \left((\hat{\partial}^\mu_{(i=0)}) + (\hat{\partial}^\mu_{(i>0)})\right) F, \quad \mu \in \{0, \pm, 3/0\}. \tag{116}$$

Now, we can follow the same lines as in the previous sections. Thus, solutions to the difference equations

$$\hat{\partial}^\mu \triangleright F = f, \quad \mu \in \{0, \pm, 3/0\} \tag{117}$$

are given by

$$F = (\hat{\partial}^\mu)^{-1} \triangleright f \tag{118}$$

$$= \sum_{k=0}^{\infty}(-1)^k \left[(\hat{\partial}^\mu_{(i=0)})^{-1}(\hat{\partial}^\mu_{(i>0)})\right]^k (\hat{\partial}^\mu_{(i=0)})^{-1}f,$$

$$\mu \in \{0, \pm, 3/0\}.$$

The operators we have to insert in (118) for evaluating can in explicit terms be written as

$$(\hat{\partial}^{3/0}_{(i=0)})_{\mathrm{L}}^{-1}f(x^+, x^{3/0}, x^3, x^-) = (D^3_{q-q^2})^{-1}f(q^{-2}x^+),$$

$$(\hat{\partial}^-_{(i=0)})^{-1}_{\mathrm{L}} f(x^+, x^{3/0}, x^3, x^-) = -q(D^+_{q^2})^{-1} f, \qquad (119)$$

$$(\hat{\partial}^+_{(i=0)})^{-1}_{\mathrm{L}} f(x^+, x^{3/0}, x^3, x^-) = -q^{-1}(D^-_{q^2})^{-1} f(q^{-2} q_+ x^3),$$

$$(\hat{\partial}^0_{(i=0)})^{-1}_{\mathrm{L}} f(x^+, x^{3/0}, x^3, x^-) = (D^{3/0}_{q^2})^{-1} f(q_- x^3)$$

and

$$(\hat{\partial}^{3/0}_{(i>0)})_{\mathrm{L}} f(x^+, x^{3/0}, x^3, x^-) \qquad (120)$$

$$= \sum_{l=1}^{\infty} \alpha^l_+ \sum_{0 \le i+j \le l} (M^-)^l_{i,j}(\underline{x}) (T^3)^i_j f,$$

$$(\hat{\partial}^-_{(i>0)})_{\mathrm{L}} f(x^+, x^{3/0}, x^3, x^-) \qquad (121)$$

$$= \frac{\lambda}{\lambda_+} \sum_{l=0}^{\infty} \alpha^l_+ \sum_{0 \le i+j \le l} \left\{ (M^+)^k_{i,j}(\underline{x}) (T^-_1)^i_j f \right.$$

$$\left. + q^{-1} (M^-)^k_{i,j}(\underline{x}) (T^-_2)^i_j f \right\},$$

$$(\hat{\partial}^+_{(i>0)})_{\mathrm{L}} f(x^+, x^3, x^{3/0}, x^-) \qquad (122)$$

$$= -q\lambda x^+ (D^{3/0}_{q^2} D^3_{q^2} f)^{-1} f(q_+ x^3)$$

$$- q \sum_{k=1}^{\infty} \alpha^k_+ \sum_{0 \le i+j \le k} \left\{ (M^-)^k_{i,j}(\underline{x}) (T^+_1)^i_j f \right.$$

$$\left. + \lambda (M^+)^k_{i,j}(\underline{x}) (T^+_2)^i_j f \right\}$$

$$- q \frac{\lambda}{\lambda_+} \sum_{0 \le k+l < \infty} \alpha^{k+l}_+$$

$$\cdot \sum_{i=0}^{k} \sum_{j=0}^{l} \sum_{0 \le u \le i+j} (M^{+-})^{k,l}_{i,j,u}(\underline{x}) (T^+_3)^{k,l}_u f$$

$$- \frac{\lambda}{\lambda_+} \sum_{0 \le k+l < \infty} \alpha^{k+l+1}_+$$

$$\cdot \sum_{i=0}^{k} \sum_{j=0}^{l+1} \sum_{0 \le u \le i+j} (M^{+-})^{k,l+1}_{i,j,u}(\underline{x}) (T^+_4)^{k,l}_u f,$$

$$(\hat{\partial}^0_{(i>0)})_{\mathrm{L}} f(x^+, x^{3/0}, x^3, x^-) \qquad (123)$$

$$= -q^2 \frac{\lambda}{\lambda_+} q^2 x^- D^-_{q^{-2}} (D^3_{q^{+2}} f)(q^2 x^{3/0}, q^2 q_+ x^3)$$

$$- q^2 \frac{\lambda}{\lambda_+} x^+ D^+_{q^2} D^3_{q^2 q_-} f$$

$$- q \frac{\lambda}{\lambda_+} x^{3/0} D^+_{q^2} D^-_{q^2} f(q^2 q_- x^3)$$

$$- q^3 \frac{\lambda^2}{\lambda_+} x^+ x^{3/0} D^+_{q^2} (D^{3/0}_{q^2} D^3_{q^2} f)(q_+ x^3)$$

$$+ \sum_{k=1}^{\infty} \alpha^k_+ \sum_{0 \le i+j \le k} \left\{ (M^+)^k_{i,j}(\underline{x}) (T^0_1)^i_j f \right.$$

$$\left. + q \frac{\lambda}{\lambda_+} (M^-)^k_{i,j}(\underline{x}) (T^0_2)^i_j f \right\}$$

$$- q^2 \frac{\lambda}{\lambda_+} \sum_{0 \le k+l < \infty} \alpha^{k+l}_+$$

$$\cdot \sum_{i=0}^{k} \sum_{j=0}^{l} \sum_{0 \le u \le i+j} (M^{+-})^{k,l}_{i,j,u}(\underline{x}) (T^0_3)^{k,l}_u f$$

$$+ \frac{\beta}{\lambda_+} \sum_{0 \le k+l < \infty} \alpha^{k+l+1}_+$$

$$\cdot \sum_{i=0}^{k+1} \sum_{j=0}^{l} \sum_{0 \le u \le i+j} (M^{+-})^{k+1,l}_{i,j,u}(\underline{x}) (T^0_4)^{k,l}_u f$$

$$+ \lambda^{-1}_+ \sum_{0 \le k+l < \infty} \alpha^{k+l+1}_+$$

$$\cdot \sum_{i=0}^{k} \sum_{j=0}^{l+1} \sum_{0 \le u \le i+j} (M^{+-})^{k,l+1}_{i,j,u}(\underline{x}) (T^+_5)^{k,l}_u f,$$

where for the purpose of abbreviation we have used the operators [10]

$$(T^{3/0})^i_j f = \left[(O^{3/0})_i f \big|_{x^3 \to x^0 + x^{3/0}} \right] (q^{2j} x^{3/0}), \qquad (124)$$

$$(T^-_1)^i_j f = \left[(O^-_1)_i f \big|_{x^3 \to x^0 + x^{3/0}} \right] (q^{2(j+1)} x^{3/0}), \quad (125)$$

$$(T^-_2)^i_j f = \left[(O^-_2)_i f \big|_{x^3 \to x^0 + x^{3/0}} \right] (q^{2j} x^{3/0}),$$

$$(T^0_1)^i_j f = \left[(O^0_1)_i f \big|_{x^3 \to y_+} - q^2 \frac{\lambda}{\lambda_+} (O^0_2)_i f \big|_{x^3 \to q^2 y_+} \right]$$
$$\cdot (q^{2j} x^{3/0}), \qquad (126)$$

$$(T^0_2)^i_j f = \left[(O^0_3)_i f \big|_{x^3 \to x^0 + x^{3/0}} + (O^0_4)_i f \big|_{x^3 \to q^2 y_-} \right]$$
$$\cdot (q^{2j} x^{3/0}),$$

$$(T^0_3)^{k,l}_u f = \left[(Q^0_1)_{k,l} f \big|_{x^3 \to x^0 + x^{3/0}} \right] (q^{2u} x^{3/0}),$$

$$(T^0_4)^{k,l}_u f = \left[(Q^0_2)_{k,l} f \big|_{x^3 \to x^0 + x^{3/0}} \right] (q^{2u} x^{3/0}),$$

$$(T^0_5)^{k,l}_u f = \left[(Q^0_3)_{k,l} f \big|_{x^3 \to x^0 + x^{3/0}} \right] (q^{2u} x^{3/0}),$$

$$(T^+_1)^i_j f = \left[(O^+_1)_i f \big|_{x^3 \to q^2 y_-} \right] (q^{2j} x^{3/0}), \qquad (127)$$

$$(T^+_2)^i_j f = \left[(O^+_2)_i f \big|_{x^3 \to y_+} \right] (q^{2j} x^{3/0}),$$

$$(T^+_3)^{k,l}_u f = \left[(Q^+_1)_{k,l} f \big|_{x^3 \to x^0 + x^{3/0}} \right] (q^{2u} x^{3/0}),$$

$$(T_4^+)_u^{k,l}f = \left[(Q_2^+)_{k,l}f\big|_{x^3 \to x^0 + x^{3/0}} \right](q^{2u}x^{3/0}).$$

The operators O_i^μ and Q_i^μ, $\mu \in \{0, \pm, 3/0\}$, as well as the polynomials $(M^\pm)_{i,j}^l$ and $(M^{+-})_{i,j,u}^{l,m}$ have already been defined in [10]. Their explicit form is once again listed in Appendix A. In addition, we have set

$$\alpha = -q^2\frac{\lambda^2}{\lambda_+^2},$$

$$\beta = q + \lambda_+,$$

$$q_\pm = 1 \pm \frac{\lambda}{\lambda_+}\frac{x^{3/0}}{x^3},$$

$$y_\pm = x^0 + \frac{2q^{\pm 1}}{\lambda_+}x^{3/0}.$$

It is also important to note that the formulae (119)–(122) hold for functions only which do not explicitly depend on x^0. Due to this fact we have to apply the substitution

$$x^0 \to x^3 - x^{3/0} \qquad (128)$$

to each function the operators in (119)–(122) shall act on.

The corresponding formulae for the second differential calculus can in the usual way be obtained from the above results by the transformations

$$(\partial^\pm)^{-1}\tilde{\triangleright}f \xleftrightarrow{\substack{\pm \leftrightarrow \mp \\ q \leftrightarrow 1/q}} (\hat{\partial}^\mp)^{-1} \triangleright f, \qquad (129)$$

$$(\partial^0)^{-1}\tilde{\triangleright}f \xleftrightarrow{\substack{\pm \leftrightarrow \mp \\ q \leftrightarrow 1/q}} (\hat{\partial}^0)^{-1} \triangleright f,$$

$$(\partial^{3/0})^{-1}\tilde{\triangleright}f \xleftrightarrow{\substack{\pm \leftrightarrow \mp \\ q \leftrightarrow 1/q}} (\hat{\partial}^{3/0})^{-1} \triangleright f,$$

symbolizing a transition via the substitutions

$$x^\pm \to x^\mp, \quad q^{\pm 1} \to q^{\mp 1}, \quad \hat{n}^\pm \to -\hat{n}^\mp, \qquad (130)$$
$$D_{q^a}^\pm \to D_{q^{-a}}^\mp, \quad (D_{q^a}^\pm)^{-1} \to (D_{q^{-a}}^\mp)^{-1}.$$

Furthermore, one has to realize that the transformations in (129) change the normal ordering. This is in complete analogy to the situation for the representations of the partial derivatives in [10]. The relationship between left and right integrals is now given by

$$f \triangleleft (\partial^0)^{-1} \xleftrightarrow{+\leftrightarrow-} -q^{-4}(\hat{\partial}^0)^{-1} \triangleright f, \qquad (131)$$

$$f \triangleleft (\partial^{3/0})^{-1} \xleftrightarrow{+\leftrightarrow-} -q^{-4}(\hat{\partial}^{3/0})^{-1} \triangleright f,$$

$$f \triangleleft (\partial^\pm)^{-1} \xleftrightarrow{+\leftrightarrow-} -q^{-4}(\hat{\partial}^\mp)^{-1} \triangleright f,$$

$$f \tilde{\triangleleft} (\hat{\partial}^0)^{-1} \xleftrightarrow{+\leftrightarrow-} -q^4(\partial^0)^{-1} \tilde{\triangleright} f, \qquad (132)$$

$$f \tilde{\triangleleft} (\hat{\partial}^{3/0})^{-1} \xleftrightarrow{+\leftrightarrow-} -q^4(\partial^{3/0})^{-1} \tilde{\triangleright} f,$$

$$f \tilde{\triangleleft} (\hat{\partial}^\pm)^{-1} \xleftrightarrow{+\leftrightarrow-} -q^4(\partial^\mp)^{-1} \tilde{\triangleright} f,$$

where the symbol $\xleftrightarrow{+\leftrightarrow-}$ has the same meaning as in Sect. 2.

Now we want to deal with the rules for integration by parts which in the case of left integrals turn out to be

$$(\partial^{3/0})_L^{-1}(\partial_L^{3/0}f) \star g\big|_{x^3=0}^a \qquad (133)$$

$$= f \star g\,\|_{x^3=0}^a - (\partial^{3/0})_L^{-1}(\Lambda^{1/2}\tau^1 f) \star \partial_L^{3/0}g\big|_{x^3=0}^a$$

$$+ q^{1/2}\lambda_+^{1/2}\lambda(\partial^{3/0})_L^{-1}(\Lambda^{1/2}(\tau^3)^{-1/2}S^1 f) \star \partial_L^+ g\big|_{x^3=0}^a,$$

$$(\partial^+)_L^{-1}(\partial_L^+ f) \star g\big|_{x^-=0}^a \qquad (134)$$

$$= f \star g\,\|_{x^-=0}^a - (\partial^+)_L^{-1}(\Lambda^{1/2}(\tau^3)^{-1/2}\sigma^2 f) \star \partial_L^+ g\big|_{x^-=0}^a$$

$$+ q^{3/2}\lambda_+^{-1/2}\lambda(\partial^+)_L^{-1}(\Lambda^{1/2}T^2 f) \star \partial_L^{3/0}g\big|_{x^-=0}^a,$$

$$(\partial^-)_L^{-1}(\partial_L^- f) \star g\big|_{x^+=0}^a \qquad (135)$$

$$= f \star g\,\|_{x^+=0}^a - (\partial^-)_L^{-1}(\Lambda^{1/2}(\tau^3)^{1/2}\tau^1 f) \star \partial_L^- g\big|_{x^+=0}^a$$

$$+ q^{-1/2}\lambda_+^{1/2}\lambda(\partial^-)_L^{-1}(\Lambda^{1/2}S^1 f) \star \partial_L^0 g\big|_{x^+=0}^a$$

$$+ \lambda^2(\partial^-)_L^{-1}(\Lambda^{1/2}(\tau^3)^{-1/2}T^- S^1 f) \star \partial_L^+ g\big|_{x^+=0}^a$$

$$- q^{-1/2}\lambda_+^{-1/2}\lambda(\partial^-)_L^{-1}(\Lambda^{1/2}(\tau^1 T^- - q^{-1}S^1)f)$$

$$\star \partial_L^{3/0}g\big|_{x^+=0}^a,$$

$$(\partial^0)_L^{-1}(\partial_L^0 f) \star g\big|_{x^{3/0}=0}^a \qquad (136)$$

$$= f \star g\,\|_{x^{3/0}=0}^a - (\partial^0)_L^{-1}(\Lambda^{1/2}\sigma^2 f) \star \partial_L^0 g\big|_{x^{3/0}=0}^a$$

$$+ q^{1/2}\lambda_+^{-1/2}\lambda(\partial^0)_L^{-1}(\Lambda^{1/2}T^2(\tau^3)^{1/2}f) \star \partial_L^- g\big|_{x^{3/0}=0}^a$$

$$- q^{-1/2}\lambda_+^{-1/2}\lambda(\partial^0)_L^{-1}(\Lambda^{1/2}(\tau^3)^{-1/2}(T^-\sigma^2 + qS^1)f)$$

$$\star \partial_L^+ g\big|_{x^{3/0}=0}^a$$

$$+ \lambda_+^{-1}(\partial^0)_L^{-1}(\Lambda^{1/2}(\lambda_-^2 T^- T^2 + q(\tau^1 - \sigma^2))f)$$

$$\star \partial_L^{3/0}g\big|_{x^{3/0}=0}^a.$$

In a similar way we get for the conjugated left integrals the identities

$$(\hat{\partial}^{3/0})_L^{-1}(\hat{\partial}_L^{3/0}f) \star g\big|_{x^3=0}^a \qquad (137)$$

$$= f \star g\,\|_{x^3=0}^a$$

$$- (\hat{\partial}^{3/0})_L^{-1}(\Lambda^{-1/2}(\tau^3)^{-1/2}\sigma^2 f) \star \hat{\partial}_L^{3/0}g\big|_{x^3=0}^a$$

$$+ q^{3/2}\lambda_+^{1/2}\lambda(\hat{\partial}^{3/0})_L^{-1}(\Lambda^{-1/2}T^2 f) \star \hat{\partial}_L^- g\big|_{x^3=0}^a,$$

$$(\hat{\partial}^-)_L^{-1}(\hat{\partial}_L^- f) \star g\big|_{x^+=0}^a \qquad (138)$$

$$= f \star g\,\|_{x^3=0}^a - (\hat{\partial}^-)_L^{-1}(\Lambda^{-1/2}\tau^1 f) \star \hat{\partial}_L^- g\big|_{x^+=0}^a$$

$$+q^{1/2}\lambda_+^{-1/2}\lambda(\hat{\partial}^-)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}(\tau^3)^{-1/2}S^1 f)$$

$$\star\;\hat{\partial}_{\mathrm{L}}^{3/0}g\Big|_{x^+=0}^a,$$

$$(\hat{\partial}^+)_{\mathrm{L}}^{-1}(\hat{\partial}_{\mathrm{L}}^+ f)\star g\Big|_{x^-=0}^a \tag{139}$$

$$= f\star g\,\|_{x^-=0}^a - (\hat{\partial}^+)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}\sigma^2 f)\star\hat{\partial}_{\mathrm{L}}^+ g\Big|_{x^-=0}^a$$

$$+q^{1/2}\lambda_+^{1/2}\lambda(\hat{\partial}^+)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}T^2(\tau^3)^{1/2}f)\star\hat{\partial}_{\mathrm{L}}^0 g\Big|_{x^-=0}^a$$

$$+q^{1/2}\lambda_+^{-1/2}\lambda(\hat{\partial}^+)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}(\tau^3)^{-1/2}(T^+\sigma^2+q\tau^3 T^2)f)$$

$$\star\;\hat{\partial}_{\mathrm{L}}^{3/0}g\Big|_{x^-=0}^a$$

$$-q^2\lambda^2(\hat{\partial}^+)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}T^2T^+ f)\star\hat{\partial}_{\mathrm{L}}^- g\Big|_{x^-=0}^a,$$

$$(\hat{\partial}^0)_{\mathrm{L}}^{-1}(\hat{\partial}_{\mathrm{L}}^0 f)\star g\Big|_{x^{3/0}=0}^a \tag{140}$$

$$= f\star g\,\|_{x^{3/0}=0}^a - (\hat{\partial}^0)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}(\tau^3)^{1/2}\tau^1 f)\star\hat{\partial}_{\mathrm{L}}^0 g\Big|_{x^{3/0}=0}^a$$

$$+q^{-1/2}\lambda_+^{-1/2}\lambda(\hat{\partial}^0)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}S^1 f)\star\hat{\partial}_{\mathrm{L}}^+ g\Big|_{x^{3/0}=0}^a$$

$$+q^{1/2}\lambda_+^{-1/2}\lambda(\hat{\partial}^0)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}(qT^+\tau^1-T^2)f)$$

$$\star\;\hat{\partial}_{\mathrm{L}}^- g\Big|_{x^{3/0}=0}^a$$

$$-\;\lambda_+^{-1}(\hat{\partial}^0)_{\mathrm{L}}^{-1}(\Lambda^{-1/2}(\tau^3)^{-1/2}$$

$$\cdot(\lambda^2 T^+S^1+q^{-1}(\tau^3\tau^1-\sigma^2))f)\star\hat{\partial}_{\mathrm{L}}^{3/0}g\Big|_{x^{3/0}=0}^a.$$

In the case of right integrals the rules for integration by parts take the form

$$(\partial^{3/0})_{\mathrm{R}}^{-1}f\star(\partial_{\mathrm{R}}^{3/0}g)\Big|_{x^3=0}^a \tag{141}$$

$$= f\star g\,\|_{x^3=0}^a - (\partial^{3/0})_{\mathrm{R}}^{-1}\partial_{\mathrm{R}}^{3/0}f\star(\Lambda^{-1/2}\sigma^2 g)\Big|_{x^3=0}^a$$

$$-\;q^{1/2}\lambda_+^{1/2}\lambda(\partial^{3/0})_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^+ f)\star(\Lambda^{-1/2}S^1 g)\Big|_{x^3=0}^a,$$

$$(\partial^+)_{\mathrm{R}}^{-1}f\star(\partial_{\mathrm{R}}^+ g)\Big|_{x^-=0}^a \tag{142}$$

$$= f\star g\,\|_{x^-=0}^a - (\partial^+)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^+ f)\star(\Lambda^{-1/2}(\tau^3)^{1/2}\tau^1 g)\Big|_{x^-=0}^a$$

$$-\;q^{3/2}\lambda_+^{-1/2}\lambda(\partial^+)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^{3/0}f)$$

$$\star\;(\Lambda^{-1/2}(\tau^3)^{1/2}T^2 g)\Big|_{x^-=0}^a,$$

$$(\partial^-)_{\mathrm{R}}^{-1}f\star(\partial_{\mathrm{R}}^- g)\Big|_{x^+=0}^a \tag{143}$$

$$= f\star g\,\|_{x^+=0}^a$$

$$-(\partial^-)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^- f)\star(\Lambda^{-1/2}(\tau^3)^{-1/2}\sigma^2 g)\Big|_{x^+=0}^a$$

$$-\;q^{3/2}\lambda_+^{1/2}\lambda(\partial^-)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^0 f)\star(\Lambda^{-1/2}(\tau^3)^{-1/2}S^1 g)\Big|_{x^+=0}^a$$

$$-\;q^2\lambda^2(\partial^-)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^+ f)\star(\Lambda^{-1/2}(\tau^3)^{-1/2}T^-S^1 g)\Big|_{x^+=0}^a$$

$$-\;q^{1/2}\lambda_+^{-1/2}\lambda(\partial^-)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^{3/0}f)$$

$$\star\;(\Lambda^{-1/2}(\tau^3)^{-1/2}(S^1-qT^-\sigma^2)g)\Big|_{x^+=0}^a,$$

$$(\partial^0)_{\mathrm{R}}^{-1}f\star(\partial_{\mathrm{R}}^0 g)\Big|_{x^{3/0}=0}^a \tag{144}$$

$$= f\star g\,\|_{x^{3/0}=0}^a - (\partial^0)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^0 f)\star(\Lambda^{-1/2}\tau^1 g)\Big|_{x^{3/0}=0}^a$$

$$-\;q^{1/2}\lambda_+^{-1/2}\lambda(\partial^0)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^- f)\star(\Lambda^{-1/2}T^2 g)\Big|_{x^{3/0}=0}^a$$

$$-\;q^{1/2}\lambda_+^{-1/2}\lambda(\partial^0)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^+ f)$$

$$\star\;(\Lambda^{-1/2}(qS^1-\tau^1T^-)g)\Big|_{x^{3/0}=0}^a$$

$$+\;\lambda_+^{-1}(\partial^0)_{\mathrm{R}}^{-1}(\partial_{\mathrm{R}}^3 f)$$

$$\star\;\Lambda^{-1/2}(\lambda^2 T^2T^-+q(\sigma^2-\tau^1))g\Big|_{x^{3/0}=0}^a.$$

Analogous formulae hold for the conjugated right integrals, as we can write

$$(\hat{\partial}^{3/0})_{\mathrm{R}}^{-1}f\star(\hat{\partial}_{\mathrm{R}}^{3/0}g)\Big|_{x^3=0}^a \tag{145}$$

$$= f\star g\,\|_{x^3=0}^a - (\hat{\partial}^{3/0})_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^{3/0}g)\star(\Lambda^{1/2}(\tau^3)^{1/2}\tau^1 g)\Big|_{x^3=0}^a$$

$$-\;q^{3/2}\lambda_+^{1/2}\lambda(\hat{\partial}^{3/0})_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^- f)\star(\Lambda^{1/2}(\tau^3)^{1/2}T^2 g)\Big|_{x^3=0}^a,$$

$$(\hat{\partial}^-)_{\mathrm{R}}^{-1}f\star(\hat{\partial}_{\mathrm{R}}^- g)\Big|_{x^+=0}^a \tag{146}$$

$$= f\star g\,\|_{x^+=0}^a - (\hat{\partial}^-)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^- f)\star(\Lambda^{1/2}\sigma^2 g)\Big|_{x^+=0}^a$$

$$-\;q^{1/2}\lambda_+^{-1/2}\lambda(\hat{\partial}^-)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^{3/0}f)\star(\Lambda^{1/2}S^1 g)\Big|_{x^+=0}^a,$$

$$(\hat{\partial}^+)_{\mathrm{R}}^{-1}f\star(\hat{\partial}_{\mathrm{R}}^+ g)\Big|_{x^-=0}^a \tag{147}$$

$$= f\star g\,\|_{x^-=0}^a - (\hat{\partial}^+)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^+ f)\star(\Lambda^{1/2}\tau^1 g)\Big|_{x^-=0}^a$$

$$-\;q^{1/2}\lambda_+^{1/2}\lambda(\hat{\partial}^+)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^{3/0}f)$$

$$\star\;(\Lambda^{1/2}(\tau^1T^++q^{-1}\tau^3 T^2)g)\Big|_{x^-=0}^a$$

$$-\;\lambda^2(\hat{\partial}^+)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^- f)\star(\Lambda^{1/2}T^2T^+ g)\Big|_{x^-=0}^a,$$

$$(\hat{\partial}^0)_{\mathrm{R}}^{-1}f\star(\hat{\partial}_{\mathrm{R}}^0 g)\Big|_{x^{3/0}=0}^a \tag{148}$$

$$= f \star g \left\|_{x^{3/0}=0}^{a} - (\hat{\partial}^0)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^0 f) \star (\Lambda^{1/2}(\tau^3)^{1/2}\sigma^2 g)\right|_{x^{3/0}=0}^{a}$$

$$- q^{3/2}\lambda_+^{-1/2}\lambda(\hat{\partial}^0)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^+ f)$$

$$\star (\Lambda^{1/2}(\tau^3)^{-1/2}S^1 g)\Big|_{x^{3/0}=0}^{a}$$

$$+ q^{-1/2}\lambda_+^{-1/2}\lambda(\hat{\partial}^0)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^- f)$$

$$\star (\Lambda^{1/2}(\tau^3)^{-1/2}(q\tau^3 T^2 - \sigma^2 T^+)g)\Big|_{x^{3/0}=0}^{a}$$

$$- q^{-1}\lambda_+^{-1}\lambda(\hat{\partial}^0)_{\mathrm{R}}^{-1}(\hat{\partial}_{\mathrm{R}}^{3/0} f)$$

$$\star (\Lambda^{1/2}(\tau^3)^{1/2}(q^{-1}\lambda^2 S^1 T^+ + (\tau^3)^{-1}\sigma^2 - \tau^2)g)\Big|_{x^{3/0}=0}^{a}.$$

What remains is to deal with the integration over the entire q-Minkowski space. In this case it is sufficient to consider the classical contributions in formula (118), as all of the other terms depending on λ lead to surface terms vanishing at infinity, if we require

$$\lim_{x^\mu \to \pm\infty} f(\underline{x}) = 0, \qquad \mu \in \{0, \pm, 3/0\},$$

$$\lim_{x^3 \to \pm\infty} (x^3)^n \left(\frac{\partial}{\partial x^3}\right)^m f(\underline{x}) = 0, \qquad \forall n, m \in \mathbb{N}. \quad (149)$$

Thus, we can write

$$(\hat{\partial}^0)^{-1}(\hat{\partial}^+)^{-1}(\hat{\partial}^-)^{-1}(\hat{\partial}^{3/0})^{-1} \rhd f \qquad (150)$$

$$= (\hat{\partial}^0_{(\mu=0)})^{-1}(\hat{\partial}^+_{(\mu=0)})^{-1}(\hat{\partial}^-_{(\mu=0)})^{-1}(\hat{\partial}^{3/0}_{(\mu=0)})^{-1} f + \mathrm{S.T.}$$

$$= (D_{q^2}^{3/0})^{-1}(D_{q^2}^-)^{-1}(D_{q^2}^+)^{-1}\left((D_{q-q^2}^3)^{-1} f\right)$$

$$\cdot (q^{-2}x^+, q^{-2}x^3) + \mathrm{S.T.}$$

where S.T. stands for neglected surface terms. The above relations follow from the same reasonings we have already applied to the Euclidean spaces. Unfortunately, the complexity of our results makes this a rather laborious task. The different possibilities for composition of the single integrals are now related to each other by the following identities:

$$(\partial^+)^{-1}(\partial^{3/0})^{-1}(\partial^-)^{-1}(\partial^0)^{-1} \rhd f \qquad (151)$$

$$= q^{-4}(\partial^-)^{-1}(\partial^{3/0})^{-1}(\partial^+)^{-1}(\partial^0)^{-1} \rhd f$$

$$= q^{-2}(\partial^{3/0})^{-1}(\partial^-)^{-1}(\partial^+)^{-1}(\partial^0)^{-1} \rhd f$$

$$= q^{-2}(\partial^{3/0})^{-1}(\partial^+)^{-1}(\partial^-)^{-1}(\partial^0)^{-1} \rhd f$$

$$= q^{-2}(\partial^-)^{-1}(\partial^+)^{-1}(\partial^{3/0})^{-1}(\partial^0)^{-1} \rhd f$$

$$= q^{-2}(\partial^+)^{-1}(\partial^-)^{-1}(\partial^{3/0})^{-1}(\partial^0)^{-1} \rhd f.$$

As we are used to, integrals over the whole space and their conjugated versions depend on each other by the transformation rules

$$(\hat{\partial}^0)^{-1}(\hat{\partial}^+)^{-1}(\hat{\partial}^-)^{-1}(\hat{\partial}^{3/0})^{-1} \rhd f \qquad (152)$$

$$\overset{\pm \leftrightarrow \mp}{\underset{q \leftrightarrow 1/q}{\longleftrightarrow}} (\partial^{3/0})^{-1}(\partial^-)^{-1}(\partial^+)^{-1}(\partial^0)^{-1} \tilde{\rhd} f.$$

If we want to have integrals over the whole space which are built up by composition of right integrals we can apply the rules

$$f \lhd (\partial^{3/0})^{-1}(\partial^-)^{-1}(\partial^+)^{-1}(\partial^0)^{-1} \qquad (153)$$

$$\overset{+ \leftrightarrow -}{\longleftrightarrow} (\bar{\partial}^0)^{-1}(\bar{\partial}^+)^{-1}(\bar{\partial}^-)^{-1}(\bar{\partial}^{3/0})^{-1} \rhd f,$$

$$f \tilde{\lhd} (\bar{\partial}^{3/0})^{-1}(\bar{\partial}^-)^{-1}(\bar{\partial}^+)^{-1}(\bar{\partial}^0)^{-1} \qquad (154)$$

$$\overset{+ \leftrightarrow -}{\longleftrightarrow} (\partial^0)^{-1}(\partial^+)^{-1}(\partial^-)^{-1}(\partial^{3/0})^{-1} \tilde{\rhd} f,$$

where $(\bar{\partial}^\mu)^{-1} = -q^{-4}(\hat{\partial}^\mu)^{-1}$.

Finally, let us come to formulae expressing right and left invariance of our integrals over the whole space. For this task we shall find it convenient to introduce the following notation:

$$\int_{\mathrm{L}} d_q V f \equiv (\partial^0)^{-1}(\partial^+)^{-1}(\partial^-)^{-1}(\partial^{3/0})^{-1} \tilde{\rhd} f,$$

$$\int_{\mathrm{R}} d_q V f \equiv f \lhd (\partial^{3/0})^{-1}(\partial^-)^{-1}(\partial^+)^{-1}(\partial^0)^{-1},$$

$$\int_{\mathrm{L}} d_q \overline{V} f \equiv (\bar{\partial}^0)^{-1}(\bar{\partial}^+)^{-1}(\bar{\partial}^-)^{-1}(\bar{\partial}^{3/0})^{-1} \rhd f,$$

$$\int_{\mathrm{R}} d_q \overline{V} f \equiv f \tilde{\lhd} (\bar{\partial}^{3/0})^{-1}(\bar{\partial}^-)^{-1}(\bar{\partial}^+)^{-1}(\bar{\partial}^0)^{-1}.$$

$$(155)$$

Right and left invariance can then be written as

$$\partial^\mu \tilde{\rhd} \int_{\mathrm{L/R}} d_q V f = \int_{\mathrm{L/R}} d_q V \partial^\mu \tilde{\rhd} f = \varepsilon(\partial^\mu) \int_{\mathrm{L/R}} d_q V f$$

$$= 0, \qquad (156)$$

$$T \rhd \int_{\mathrm{L/R}} d_q V f = \int_{\mathrm{L/R}} d_q V T \rhd f = \varepsilon(T) \int_{\mathrm{L/R}} d_q V f$$

$$= 0,$$

$$K \rhd \int_{\mathrm{L/R}} d_q V f = \int_{\mathrm{L/R}} d_q V K \rhd f = \varepsilon(K) \int_{\mathrm{L/R}} d_q V f$$

$$= \int_{\mathrm{L/R}} d_q V f,$$

$$\left(\int_{\mathrm{L/R}} d_q V f\right) \lhd \partial^\mu = \int_{\mathrm{L/R}} d_q V f \lhd \partial^\mu = \varepsilon(\partial^\mu) \int_{\mathrm{L/R}} d_q V f$$

$$= 0, \qquad (157)$$

$$\left(\int_{\mathrm{L/R}} d_q V f\right) \lhd T = \int_{\mathrm{L/R}} d_q V f \lhd T = \varepsilon(T) \int_{\mathrm{L/R}} d_q V f$$

$$= 0,$$

$$\left(\int_{\text{L/R}} d_q V f \right) \triangleleft K = \int_{\text{L/R}} d_q V f \triangleleft K = \varepsilon(K) \int_{\text{L/R}} d_q V f$$

$$= \int_{\text{L/R}} d_q V f,$$

where T and K denote Lorentz generators from one of the following two sets:

$$T \in \{T^{\pm}, S^1, T^2\}, \qquad K \in \{\tau^3, \tau^1, \sigma^2\}. \tag{158}$$

Anlogous identities hold for the conjugated integrals.

5 Remarks

In the past several attempts have been made to introduce the notion of integration on quantum spaces [22–26]. However, our way of introducing integration is much more in the spirit of [27], where integration has also been considered as an inverse of q-differentiation. While in [27] the algebraic point of view dominates, our integration formulae are directly derived by inversion of q-difference operators. Thus, this method of integration represents nothing else than a generalization of Jackson's celebrated q-integral to higher dimensions.

Let us end with a few comments on some typical features of our integrals. Their expressions are affected by the non-commutative structure of the underlying quantum spaces in two ways. First of all, integration can always be reduced to a process of summation. Additionally, there is a correction term for each order of λ vanishing in the undeformed limit ($q = 1$). These new terms result from the finite boundaries of integration and are responsible for the fact that integral operators referring to different directions do not commute. However, if we integrate over the whole space, the single integrals the volume integral is composed of become independent from each other and can then be expressed by Jackson integrals, only.

One important problem which is still open at the moment concerns the question of integrability. It can easily be seen that for polynomials the series representing our integrals terminates. Thus, we can say that polynomials are always integrable. For all other functions which do not vary too rapidly compared to λ it seems to be resonable to assume that the correction terms become small enough to constitute a convergent series.

A Notation

On the notation presently nine remarks will be made.
(1) The q-*number* is defined by [13]

$$[[c]]_{q^a} \equiv \frac{1 - q^{ac}}{1 - q^a}, \qquad a, c \in \mathbb{C}. \tag{159}$$

For $m \in \mathbb{N}$, we can introduce the q-*factorial* by setting

$$[[m]]_{q^a}! \equiv [[1]]_{q^a} [[2]]_{q^a} \dots [[m]]_{q^a},$$

$$[[0]]_{q^a}! \equiv 1. \tag{160}$$

There is also a q-analogue of the usual binomial coefficients, the so-called q-*binomial coefficients* defined by the formula

$$\begin{bmatrix} \alpha \\ m \end{bmatrix}_{q^a} \equiv \frac{[[\alpha]]_{q^a} [[\alpha - 1]]_{q^a} \dots [[\alpha - m + 1]]_{q^a}}{[[m]]_{q^a}!}, \tag{161}$$

where $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$.
(2) Note that in functions only such variables are explicitly displayed as are affected by a scaling. For example, we write

$$f(q^2 x^+) \qquad \text{instead of} \qquad f(q^2 x^+, x^3, x^-). \tag{162}$$

(3) Arguments enclosed in parentheses refer to the first object on their left. For example, we have

$$D_{q^2}^+ f(q^2 x^+) = D_{q^2}^+ (f(q^2 x^+)) \tag{163}$$

or

$$D_{q^2}^+ (D_{q^2}^+ f + D_{q^2}^- f)(q^2 x^+) = D_{q^2}^+ \left[(D_{q^2}^+ f + D_{q^2}^- f)(q^2 x^+) \right]. \tag{164}$$

However, the symbol $|_{x' \to x}$ applies to the whole expression on its left side reaching up to the next opening bracket or $\pm$ sign.
(4) The *Jackson derivative* referring to the coordinate x^A is defined by

$$D_{q^a}^A f := \frac{f(x^A) - f(q^a x^A)}{(1 - q^a) x^A}, \tag{165}$$

where f may depend on other coordinates as well. Higher Jackson derivatives are obtained by applying the above operator $D_{q^a}^A$ several times:

$$\left(D_{q^a}^A \right)^i f := \underbrace{D_{q^a}^A D_{q^a}^A \dots D_{q^a}^A}_{i \text{ times}} f. \tag{166}$$

(5) For $a > 0$, $q > 1$ and $x^A > 0$, the definition of the *Jackson integral* is

$$(D_{q^a}^A)^{-1} f \Big|_0^{x^A} = -(1 - q^a) \sum_{k=1}^{\infty} (q^{-ak} x^A) f(q^{-ak} x^A),$$

$$(D_{q^a}^A)^{-1} f \Big|_{x^A}^{\infty} = -(1 - q^a) \sum_{k=0}^{\infty} (q^{ak} x^A) f(q^{ak} x^A), \tag{167}$$

$$(D_{q^{-a}}^A)^{-1} f \Big|_0^{x^A} = (1 - q^{-a}) \sum_{k=0}^{\infty} (q^{-ak} x^A) f(q^{-ak} x^A),$$

$$(D_{q^{-a}}^A)^{-1} f \Big|_{x^A}^{\infty} = (1 - q^{-a}) \sum_{k=1}^{\infty} (q^{ak} x^A) f(q^{ak} x^A).$$

For $a > 0$, $q > 1$ and $x^A < 0$, we set

$$(D_{q^a}^A)^{-1} f \Big|_{x^A}^0 = (1 - q^a) \sum_{k=1}^{\infty} (q^{-ak} x^A) f(q^{-ak} x^A),$$

$$(D_{q^a}^A)^{-1} f\big|_{-\infty}^{x^A} = (1-q^a) \sum_{k=0}^{\infty} (q^{ak} x^A) f(q^{ak} x^A), \quad (168)$$

$$(D_{q^{-a}}^A)^{-1} f\big|_{x^A}^{0} = -(1-q^{-a}) \sum_{k=0}^{\infty} (q^{-ak} x^A) f(q^{-ak} x^A),$$

$$(D_{q^{-a}}^A)^{-1} f\big|_{-\infty}^{x^A} = -(1-q^{-a}) \sum_{k=1}^{\infty} (q^{ak} x^A) f(q^{ak} x^A).$$

Note that the formulae (167) and (168) also yield expressions for q-integrals over any other interval [13].

(6) Additionally, we need operators of the following form:

$$\hat{n}^A \equiv x^A \frac{\partial}{\partial x^A}. \quad (169)$$

(7) Calculations for q-deformed Minkowski space show that it is reasonable to give the following repeatedly appearing polynomials a name of their own:

$$(S_q)_{i,j}(x^0, x^{3/0})$$

$$\equiv \begin{cases} \sum_{p_1=0}^{j} \sum_{p_2=0}^{p_1} \cdots \sum_{p_{i-j}=0}^{p_{i-j-1}} \prod_{l=0}^{i-j} a_{q^{-1}}(x^0, q^{2p_l} x^{3/0}), \\ 1, \quad \text{if } j = i, \end{cases}$$

$$a_q(x^0, x^{3/0}) \equiv -q x^{3/0}(q x^{3/0} + \lambda_+ x^0), \quad (170)$$

$$(M^\pm)_{i,j}^k(\underline{x}) \equiv (M^\pm)_{i,j}^k(x^0, x^+, x^{3/0}, x^-)$$

$$= \binom{k}{i} \lambda_+^j (a_{q^{\pm 1}}(q^{2j} x^{3/0}))^i (x^+ x^-)^j (S_q)_{k-j,j}(x^0, x^{3/0}),$$

$$(M^{+-})_{i,j,u}^{k,l}(\underline{x}) \equiv (M^{+-})_{i,j,u}^{k,l}(x^0, x^+, x^{3/0}, x^-)$$

$$= \binom{k}{i}\binom{l}{j} \lambda_+^u (x^+ x^-)^u (a_q(q^{2u} x^{3/0}))^{k-i}$$

$$\cdot (a_{q^{-1}}(q^{2u} x^{3/0}))^{l-j} (S_q)_{i+j,u}(x^0, x^{3/0}).$$

(8) In [10] we have introduced the quantities $(K_\alpha)_{a_1,\dots,a_l}^{(k_1,\dots,k_l)}$, $k_i \in \mathbb{N}$, $a_i, \alpha \in \mathbb{R}$. We also refer to [10] for a review of their explicit calculation. These quantities can be used to define new operators by setting

$$D_{a_1,\dots,a_l}^{(k_1,\dots,k_l)} x^n = \begin{cases} (K_n)_{a_1,\dots,a_l}^{(k_1,\dots,k_l)} x^{n-k_1-\dots-k_l}, \\ 0, \quad \text{if } n < k_1 + \dots + k_l, \end{cases} \quad (171)$$

Especially, in the case of q-deformed Minkowski space we need the operators

$$(D_{1,q}^3)^{k,l} \equiv D_{1,q^2}^{(k,l)}, \quad (172)$$

$$(D_{2,q}^3)^{k,l} \equiv D_{y_-/x^3,q^2 y_-/x^3}^{(k,l)},$$

$$(D_{3,q}^3)_{i,j}^{k,l} \equiv D_{y_+/x^3,q^2 y_+/x^3,y_-/x^3,q^2 y_-/x^3}^{(k,l,i,j)},$$

where

$$y_\pm = y_\pm(x^0, x^{3/0}) = x^0 + \frac{2q^{\pm 1}}{\lambda_+} x^{3/0}. \quad (173)$$

Notice that these operators have to act upon the coordinate x^3, only.

(9) In Sect. 4 the representations of $(\hat{\partial}^\mu)^{-1}$, $\mu \in \{\pm, 0, 3/0\}$, have been formulated by using the operators

$$(O^{3/0})_k f = (D_{2,q}^3)^{k,k+1} f(q^2 x^+), \quad (174)$$

$$(O_1^-)_k f = x^- (D_{2,q}^3)^{k+1,k+1} f(q^2 x^+), \quad (175)$$

$$(O_2^-)_k f = x^{3/0} D_{q^2}^+ (D_{2,q}^3)^{k,k+1} f(q^2 x^+),$$

$$(O_1^0)_k f = D_{q^2}^{3/0} (D_{1,q}^3)^{k,k} f \quad (176)$$
$$- q^3 \lambda_+^{-1} \lambda^2 x^+ x^{3/0} D_{q^2}^+ D_{q^2}^{3/0} (D_{1,q}^3)^{k,k+1} f,$$

$$(O_2^0)_k f = x^- (D_{1,q^{-1}}^3)^{k,k+1} D_{q^2}^- f(q^2 x^{3/0}),$$

$$(O_3^0)_k f = q x^+ D_{q^2}^+ (D_{2,q}^3)^{k,k+1} f,$$

$$(O_4^0)_k f = x^{3/0} D_{q^2}^+ (D_{1,q^{-1}}^3)^{k,k} D_{q^2}^- f,$$

$$(Q_1^0)_{k,l} f = (x^0 + q^{-1}\lambda x^3)(D_{3,q}^3)_{l,l+1}^{k+1,k} f \quad (177)$$
$$+ q \lambda_+^{-1} \lambda (q + \lambda_+) x^+ x^{3/0} D_{q^2}^+ (D_{3,q}^3)_{l,l+1}^{k,k+1} f$$
$$- q^3 \lambda_+^{-1} \lambda^2 x^+ x^{3/0}(x^0 + q^{-1}\lambda x^{3/0})$$
$$\cdot D_{q^2}^+ (D_{3,q}^3)_{l,l+1}^{k+1,k+1} f,$$

$$(Q_2^0)_{k,l} f = (D_{3,q}^3)_{l,l+1}^{k+1,k+1} f,$$

$$(Q_3^0)_{k,l} f = q^{-1}(D_{3,q}^3)_{l+1,l+1}^{k+1,k} f$$
$$- q^2 \lambda_+^{-1} \lambda^2 x^+ x^{3/0} D_{q^2}^+ (D_{3,q}^3)_{l+1,l+1}^{k+1,k+1} f,$$

$$(O_1^+)_k f = (D_{1,q^{-1}}^3)^{k,k} D_{q^2}^- f, \quad (178)$$

$$(O_2^+)_k f = x^+ D_{q^2}^{3/0} (D_{1,q}^3)^{k,k+1} f,$$

$$(Q_1^+)_{k,l} f = (q + \lambda_+) x^+ (D_{3,q}^3)_{l,l+1}^{k,k+1} f \quad (179)$$
$$- q^2 \lambda x^+ (x^0 + q^{-1}\lambda x^3)(D_{3,q}^3)_{l,l+1}^{k+1,k+1} f,$$

$$(Q_2^+)_{k,l} f = x^+ (D_{3,q}^3)_{l+1,l+1}^{k+1,k+1} f.$$

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